1. (6 pts. each) Consider the function \( f(z) = e^z = e^{x-iy} \).

(a) Find real functions \( u(x, y) \) and \( v(x, y) \) such that \( f(x + iy) = u(x, y) + iv(x, y) \).

(b) Is \( f(z) \) analytic? Justify your answer using the Cauchy-Riemann equations.

**Solution.**

(a).

\[
e^{x-iy} = e^x e^{-iy} = e^x (\cos(-y) + i \sin(-y)) = e^x (\cos y - i \sin y).
\]

Therefore, \( f(x + iy) = u(x, y) + iv(x, y) = e^x \cos y + i(-e^x \sin y) \).

(b). \( u_x = e^x \cos y, \ u_y = -e^x \sin y, \ v_x = -e^x \sin y, \ v_y = -e^x \cos y \). Hence \( u_x \neq v_y \) and \( u_y \neq -v_x \) so that \( f(z) \) is not analytic.

2. (6 pts. each) Evaluate the following integrals around the unit circle (clockwise) either directly or by citing an appropriate theorem. Show all work or clearly state which result you are using.

(a) \( \int_C (\bar{z} + \frac{1}{\bar{z}}) \, dz \)

(b) \( \int_C \frac{7z - 6}{z^2 - 2z} \, dz \)

**Solution.**

(a). Since \( \bar{z} + 1/\bar{z} \) is not analytic, we cannot use any of the Cauchy Theorems, so we must evaluate the integral directly. For that we parametrize the unit circle by \( z = e^{-it}, \ 0 \leq t \leq 2\pi \) (note that the minus sign takes care of the clockwise orientation of the circle). Hence \( \bar{z} = e^{it} \) and \( 1/\bar{z} = e^{-it} \), and finally \( dz = -i e^{-it} \, dt \). Therefore,

\[
\int_C \left( \bar{z} + \frac{1}{\bar{z}} \right) \, dz = \int_0^{2\pi} (e^{it} + e^{-it})(-i e^{-it}) \, dt
\]

\[
= (-i) \int_0^{2\pi} (1 + e^{-2it}) \, dt
\]

\[
= -2\pi i
\]

(b). Since the integrand is analytic we can use Cauchy’s Integral Theorem to do this integral. Note that \( z = 0 \) is inside the contour of integration while \( z = 2 \) is
\[ \int_{C} \frac{7z - 6}{z^2 - 2z} \, dz = \int_{C} \frac{7z - 6}{z(z - 2)} \, dz = \int_{C} \frac{7z - 6}{z - 2} \, \frac{dz}{z} = -2\pi i \left( \frac{7z - 6}{z - 2} \right)_{z=0} = -6\pi i. \]

Note that the minus sign is due to the fact that the contour of integration is oriented clockwise.

3. (6 pts. each) Evaluate the following integrals around the given curves.

(a) \( \int_{C} \frac{\cos 2z}{(z - \pi/2)^4} \, dz \) around the curve \(|z| = 2\) (counterclockwise).

(b) \( \int_{C} \frac{dz}{(z^2 + 2)^2} \) around the curve \(|z - i| = 1\) (counterclockwise).

Solution.

(a) Since the integrand is analytic in a domain containing the contour of integration and its interior, we can use Cauchy’s Derivative Formula to compute this integral. Note that \( z = \pi/2 \) is inside the contour of integration.

\[
\int_{C} \frac{\cos 2z}{(z - \pi/2)^4} \, dz = \frac{2\pi i}{3!} \frac{d^3}{dz^3} (\cos(2z))|_{z=\pi/2} = \frac{\pi i}{3} (8 \cos(\pi)) = 0.
\]

(b) We again use Cauchy’s Derivative Formula to compute this integral. Note that \((z^2 + 2)^2 = (z - i\sqrt{2})^2 (z + i\sqrt{2})^2\) and that \(i\sqrt{2}\) is inside the contour of integration while \(-i\sqrt{2}\) is not. Therefore

\[
\int_{C} \frac{dz}{(z^2 + 2)^2} = \int_{C} \frac{dz}{(z - i\sqrt{2})^2 (z + i\sqrt{2})^2} = \frac{2\pi i}{1!} \frac{d}{dz} (\frac{1}{(z + i\sqrt{2})^2})|_{z=i\sqrt{2}} = 2\pi i (-2)(i\sqrt{2} + i\sqrt{2})^{-3} = -4\pi i \frac{\pi}{(2i\sqrt{2})^3} = \frac{\pi}{4\sqrt{2}}.
\]
4. (6 pts. each) Find the center and radius of convergence of each of the following power series using any method you like.

(a) \[ \sum_{n=0}^{\infty} \frac{n^4}{2^n} z^{2n}. \]

(b) \[ \sum_{n=0}^{\infty} n(n-1)(z-3+2i)^n. \]

**Solution.**

(a). The center of the series is \( z_0 = 0. \)

\[
\lim_{n \to \infty} \left| \frac{(n+1)^4 z^{2n+2}}{2^{n+1}} \right| = \lim_{n \to \infty} 2^{n+1} \frac{n+1}{n} \left| z \right|^2 = 2|z|^2.
\]

Hence the series converges for \( 2|z|^2 < 1 \) or \( |z| < 1/\sqrt{2}. \) Hence \( R = 1/\sqrt{2}. \)

(b). The center of the series is \( z_0 = 3 - 2i. \)

\[
\lim_{n \to \infty} \left| \frac{(n+1)n(z-3+2i)^{n+1}}{n(n-1)(z-3+2i)^n} \right| = \lim_{n \to \infty} \frac{n+1}{n-1} |z| = |z|.
\]

Hence the series converges for \( |z| < 1 \), so that \( R = 1. \)

5. (6 pts. each) Let \( f(z) = \sum_{n=1}^{\infty} \frac{n}{2^n} (z+i)^{2n}. \)

(a) Find a power series representation for \( f''(z). \)

(b) Find a power series representation for an analytic function \( F(z) \) such that \( F''(z) = f(z). \)

**Solution.**

(a).

\[
\begin{align*}
    f(z) & = \sum_{n=1}^{\infty} \frac{n}{2^n} (z+i)^{2n} \\
    f'(z) & = \sum_{n=1}^{\infty} \frac{(2n)(n)}{2^{n+1}} (z+i)^{2n-1} \\
    f''(z) & = \sum_{n=1}^{\infty} \frac{(2n)(2n-1)(n)}{2^n} (z+i)^{2n-2}.
\end{align*}
\]
Note that since the original series started with a $z^2$ term, the final series still begins at $n = 1$.

(b). Integrating the series $f(z) = \sum_{n=1}^{\infty} \frac{n}{2^n}(z + i)^{2n}$ once gives

$$\sum_{n=1}^{\infty} \frac{n}{(2n + 1)2^n}(z + i)^{2n+1}$$

and integrating it twice gives $F(z) = \sum_{n=1}^{\infty} \frac{n}{(2n + 1)(2n + 2)2^n}(z + i)^{2n+2}$. 