1. (6 pts.) Find the volume of the parallelepiped determined by the vectors \([2, 0, 3], [0, 4, 1],\) and \([5, 6, 0].\)

Solution:

\[
Volume = |[2, 0, 3] \cdot ([0, 4, 1] \times [5, 6, 0])| \\
= |(2i + 3k) \cdot ((4j + k) \times (5i + 6j))| \\
= |(2i + 3k) \cdot (20(j \times i) + 5(k \times i) + 6(k \times j))| \\
= |(2i + 3k) \cdot (−6i + 5j − 20k)| \\
= |−72| = 72.
\]

2. (6 pts.) Find the work done by the force \(\mathbf{F} = [e^x, e^y, e^z]\) in the displacement along the straight line segment from \((0, 0, 0)\) to \((2, 4, 4).\)

Solution:

A parametrization for the straight line segment from \((0, 0, 0)\) to \((2, 4, 4)\) is given by \(\mathbf{r}(t) = [2t, 4t, 4t], 0 ≤ t ≤ 1.\) Therefore

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\
= \int_0^1 [e^{2t}, e^{4t}, e^{4t}] \cdot [2, 4, 4] dt \\
= \int_0^1 2e^{2t} + 8e^{4t} dt \\
= e^{2t} + 2e^{4t} \bigg|_0^1 = 2e^2 + e^4 - 3.
\]

3. (6 pts. each) Let \(f = xz^2 + yz^2 + yx^2.\)

(a) Verify for the given function \(f\) that \(\text{curl}(\nabla f) = 0.\)

(b) Find the rate of change of \(f\) at the point \((1, 1, −1)\) in the direction of the vector \([4, 0, 3].\)

Solution:

(a). Note that \(\nabla f = [z^2 + 2xy, x^2 + 2yz, y^2 + 2xz]\) so that

\[
\text{curl}(\nabla f) = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
z^2 + 2xy & x^2 + 2yz & y^2 + 2xz
\end{vmatrix} \\
= \mathbf{i}(2y - 2y) - \mathbf{j}(2z - 2z) + \mathbf{k}(2x - 2x) = 0.
\]
(b). The unit vector in the direction of \([4, 0, 3]\) is \(\mathbf{u} = \left[\frac{4}{5}, 0, \frac{3}{5}\right]\). Therefore the rate of change of \(f\) at the point \((1, 1, -1)\) in the direction \(\mathbf{u}\) is

\[
D_{\mathbf{u}} f(1, 1, -1) = \nabla f(1, 1, -1) \cdot \mathbf{u} = \left[3, -1, -1\right] \cdot \left[\frac{4}{5}, 0, \frac{3}{5}\right] = \frac{9}{5}.
\]

4. (6 pts.) Given that the differential is exact, evaluate the integral \(\int_{C} z e^{xz} \, dx + dy + x e^{xz} \, dz\) where \(C\) is a curve beginning at the point \((2, 3, 0)\) and ending at the point \((0, 1, 2)\).

Solution:

Since the differential \(z e^{xz} \, dx + dy + x e^{xz} \, dz\) is exact we must find a function \(f\) such that \(df = z e^{xz} \, dx + dy + x e^{xz} \, dz\). Since \(\partial f/\partial x = z e^{xz}\), \(f = e^{xz} + g(y, z)\). Since \(\partial f/\partial y = 1\), \(\partial g/\partial y = 1\), so that \(g(y, z) = y + h(z)\). Since \(\partial f/\partial z = x e^{xz}\), \(h'(z) = 0\), so that \(h(z)\) is constant. Choosing that constant to be zero gives \(f = e^{xz} + y\).

Since the vector field is conservative, the line integral is independent of path, so that

\[
\int_{C} z e^{xz} \, dx + dy + x e^{xz} \, dz = f(0, 1, 2) - f(2, 3, 0) = 2 - 4 = -2.
\]

5. (6 pts. each) Let \(S\) be the surface of the paraboloid given by the equation \(z = x^2 + y^2\), \(0 \leq z \leq 4\), and let \(\mathbf{r}(u, v) = [u \cos v, u \sin v, u^2]\), \(0 \leq u \leq 2\), \(0 \leq v \leq 2\pi\) be a parametrization of \(S\).

(a) Find a normal vector \(\mathbf{N}\) of the surface \(S\).

(b) Set up but do not evaluate an iterated double integral for the surface integral \(\iint_{S} xyz \, dA\). (Hint: By “set up” I mean do all the work as though you were actually computing the integral, up to the point where you would begin computing antiderivatives, then stop.)

(c) Set up but do not evaluate an iterated double integral for the flux \(\iint_{S} \mathbf{F} \cdot \mathbf{n} \, dA\) where \(\mathbf{F} = [z, 0, x]\).

Solution:

(a). \(\mathbf{r}_u = [\cos v, \sin v, 2u]\), \(\mathbf{r}_v = [-u \sin v, u \cos v, 0]\) so that

\[
\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix} = \mathbf{i}(-2u^2 \cos v) - \mathbf{j}(2u^2 \sin v) + \mathbf{k}(u \cos^2 v + u \sin^2 v) = -2u^2 \cos v \mathbf{i} - 2u^2 \sin v \mathbf{j} + \mathbf{k}.
\]
(b). Here the parameter region is $R = [0, 2] \times [0, 2\pi]$. Therefore,

$$\iint_S xyz \, dA = \iint_R (u \cos v)(u \sin v)(u^2) - 2u^2 \cos v \, i - 2u^2 \sin v \, j + u k \, du \, dv$$

$$= \int_0^{2\pi} \int_0^2 u^4 \sin v \, (4u^4 \cos^2 v + 4u^4 \sin^2 v + u^2)^{1/2} \, du \, dv$$

$$= \int_0^{2\pi} \int_0^2 u^4 \sin v \sin \left(4u^4 + u^2\right)^{1/2} \, du \, dv$$

$$= \int_0^{2\pi} \int_0^2 u^4 \cos v \sin v \, (4u^4 + 1)^{1/2} \, du \, dv$$

$$= \int_0^{2\pi} \int_0^2 u^5 (4u^2 + 1)^{1/2} \cos v \sin v \, du \, dv$$

(c).

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \iint_R \mathbf{F}(r(u, v)) \cdot (r_u \times r_v) \, du \, dv$$

$$= \int_0^{2\pi} \int_0^2 \left[ u^2, 0, u \cos v \right] \cdot [-2u^2 \cos v, -2u^2 \sin v, u] \, du \, dv$$

$$= \int_0^{2\pi} \int_0^2 -2u^4 \cos v + u^2 \cos v \, du \, dv$$

$$= \int_0^{2\pi} \int_0^2 u^2 \cos v (1 - 2u^2) \, du \, dv$$

6. (6 pts.) Use the Divergence Theorem to compute the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} \, dA$ where $\mathbf{F} = [x^2, z^2, y^2]$ and $S$ is the surface of the tetrahedron with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$. (Hint: The slanted surface of the tetrahedron is the plane with equation $x + y + z = 2$.)

Solution:

The Divergence Theorem says that

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \iiint_T \text{div} \mathbf{F} \, dV$$

where $T$ is the region enclosed by $S$. In our case,

$$\text{div} \mathbf{F} = \text{div}[x^2, y^2, z^2] = 2x.$$ 

$$\iiint_T \text{div} \mathbf{F} \, dV = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2x \, dz \, dy \, dx$$

$$= \int_0^2 \int_0^{2-x} 2x(2 - x - y) \, dy \, dx$$

$$= \int_0^2 2xy(2 - x) - xy^2 \Big|_0^{2-x} \, dx$$
\[
\begin{align*}
&= \int_0^2 x(2 - x)^2 \, dx \\
&= \int_0^2 x^3 - 4x^2 + 4x \, dx \\
&= \left. \frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2 \right|_0^2 = 12 - \frac{32}{3} = \frac{4}{3}
\end{align*}
\]