MATH 290 – 1 JULY 2009 – EXAM 4

Answer all of the following questions on the answer sheets provided. Show all work, as partial credit may be given.

1. (10 pts.) Use induction to prove that for all natural numbers n with $n \geq 5$, $2^{n+2} > n^3$.

2. (10 pts.) Define the Fibonacci sequence $\{f_n\}$ as follows. $f_1 = f_2 = 1$ and for $n \in \mathbf{N}$, $f_{n+2} = f_{n+1} + f_n$. Prove that for all natural numbers n,

$$f_1 + f_2 + \cdots + f_n = f_{n+2} - 1.$$

3. (10 pts.) Consider the following theorem: *If p is prime, and if $p|x_1\, x_2 \cdots x_n$ where $x_1, x_2, \ldots, x_n \in \mathbf{N}$, then $p|x_i$ for some i.* Prove that this theorem holds for all natural numbers $n \geq 2$, assuming that it holds for $n = 2$. In other words, prove only the inductive step: *If the result holds for all $k \in \{2, 3, \ldots, n-1\}$ then it holds for n.* (Hint: $x_1\, x_2 \cdots x_n = y_1\, y_2$ where $y_1 = x_1\, x_2 \cdots x_{n-1}$ and $y_2 = x_n$.)

4. (5 pts. each) A group of 5 boys and 3 girls is to be photographed.

   (a) In how many ways can the children be arranged in one row?

   (b) In how many ways can the children be arranged in one row if the boys and the girls must stand together?

5. (10 pts.) Prove that for any nonempty sets A, B, C, and D, $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.