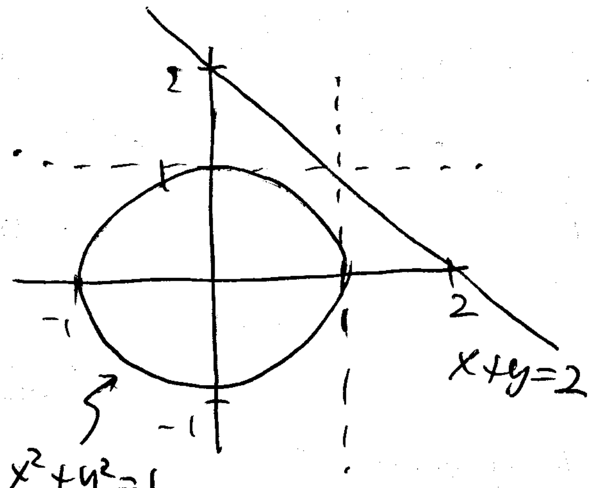
MATH 290 - EXAM 2 - SOLUTIONS

1. Let $m = 89$. Then $m \in \mathbb{N}$, $m < 100$, the sum of the digits of m is $8 + 9 = 17$, and the product of the digits of m is $8 \cdot 9 = 72$.

The point is that to prove a proposition of the form $(\exists x)\, P(x)$, it is sufficient to simply find an x such that $P(x)$ is true.

2\.



Suppose x and y are real numbers such that $x + y = 2$. We must show that $x^2 + y^2 > 1$. Look at 3 cases: (i) $x < 1$, (ii) $x = 1$, (iii) $x > 1$.

Case 1: If $x < 1$ and $x + y = 2$ then $y > 1$ so that $x^2 + y^2 \geq y^2 > 1$.

Case 2: If $x=1$ and $x+y=2$ then $y=1$.
Therefore $x^2+y^2=2>1$.
Case 3: If $x>1$ then $x^2+y^2 \geq x^2 > 1$.

3. [Proving $P \Rightarrow Q$ by contraposition means proving the equivalent proposition $\sim Q \Rightarrow \sim P$]

Suppose that x is even and y is even [this is $\sim Q$]. Then there exist integers k and j such that $x=2k$ and $y=2j$. Therefore $xy=(2k)(2j)=4kj=2(2kj)$. Since $2kj$ is an integer, xy is even [this is $\sim P$]. ☒

4. Assume that there exists an integer $k>1$ such that $k \mid (k+1)$ [this is the negation of what is to be proved]. Then there is a positive integer m such that $mk=k+1$. Therefore $k(m-1)=1$. If $m=1$ this is $0=1$, contradicting the known fact that $0 \neq 1$. If $m>1$ then $m-1 \geq 1$ so $k(m-1) > m-1 \geq 1$ since $k>1$. This means $1>1$ contradicting the known fact that $1 \leq 1$. ☒

5. We must show that if n^3 is even then n is even, and if n is even then n^3 is even. We will show the first implication by contraposition. Suppose that n is odd. Then there exists an integer k such that $n = 2k+1$. Hence $n^3 = (2k+1)^3 = (2k)^3 + 3(2k)^2 + 3(2k) + 1$
$= 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1.$
Since $4k^3 + 6k^2 + 3k$ is an integer, n^3 is odd.
Now we show the second implication.
Suppose that n is even. Then there exists an integer k such that $n = 2k$. Therefore $n^3 = (2k)^3 = 8k^3 = 2(4k^3)$. Since $4k^3$ is an integer, n^3 is even. $\boxtimes$

Alternate proofs for #2:

① Contrapositive: Suppose that $x^2+y^2 \le 1$. We must show that $x+y \ne 2$. Since $x^2+y^2 \le 1$ it must be true that $x \le 1$ and $y \le 1$. Hence $x+y \le 2$. If $x \le 1$ and $y \le 1$ then the only way that $x+y=2$ is that $x=y=1$. In this case $x^2+y^2 = 2 > 1$. Hence if $x^2+y^2 \le 1$ then $x+y < 2$.

② Algebraic approach: Suppose that ~~x~~ $x+y=2$. Then $x=2-y$ and $x^2+y^2=(2-y)^2+y^2=4-4y+2y^2$. We want to show that $x^2+y^2>1$. This is equivalent to showing $4-4y+2y^2>1$ or $2y^2-4y+3>0$. But the discriminant of this quadratic is $\sqrt{(-4)^2-4(2)(3)} = \sqrt{-8}$ so that $2y^2-4y+3=0$ has no real solutions, so is positive for all y.