Exam 4   2.4 – 3.1

3.1 2, 3c, 13a, 16

2.5 5b

~~2.4 4a~~

2.5 5(b) $GCD(f_n, f_{n+1}) = 1$ for all $n \in \mathbb{N}$

Pf: (1) If $n = 1$, $f_1 = 1$ and $f_2 = 1$, then $GCD(f_1, f_2) = 1$.

(2) Assume that for some $n \in \mathbb{N}$, $GCD(f_n, f_{n+1}) = 1$.
Want to show that $GCD(f_{n+1}, f_{n+2}) = 1$.

$f_{n+2} = f_{n+1} + f_n$

Spse $GCD(f_{n+1}, f_{n+2}) = d$ with $d > 1$

So $d | f_{n+1}$ and $d | f_{n+2}$

Then $d | (f_{n+2} - f_{n+1})$ so $d | f_n$

Hence $GCD(f_{n+1}, f_n) \geq d > 1$ contradiction

Suppose that $GCD(f_{n+1}, f_{n+2}) = d > 1$. Then $d | f_{n+1}$ and $d | f_{n+2}$ so $d | (f_{n+2} - f_{n+1})$. That is, $d | f_n$. Hence d is a common divisor of f_n and f_{n+1} so $GCD(f_n, f_{n+1}) \geq d > 1$. Hence $GCD(f_n, f_{n+1}) \neq 1$. ~~Hence~~ This contradicts our induction hypothesis, so $GCD(f_{n+1}, f_{n+2}) = 1$.

3.1 2) Let A, B be nonempty. Prove that $A \times B = B \times A$ iff $A = B$.

$A \times B = \{(a,b) : a \in A, b \in B\}$

$B \times A = \{(b,a) : b \in B, a \in A\}$.

($\Longrightarrow$) Let $a \in A$, then given $b \in B$, $(a,b) \in A \times B = B \times A$. Hence $a \in B$ so $A \subseteq B$. Similarly $B \subseteq A$.

Pf: We will just show that if $A \times B = B \times A$ then $A = B$. Let $x \in A$. Since B is not empty there is $y \in B$ and so $(x,y) \in A \times B$. Since $A \times B = B \times A$, $(x,y) \in B \times A$. Hence $x \in B$ and $A \subseteq B$. Let $y \in B$. Then since A is nonempty there is an $x \in A$ so that $(y,x) \in B \times A$. But since $B \times A = A \times B$, $y \in A$ so $B \subseteq A$. Hence $A = B$.

Q. True if $A = \emptyset$ or $B = \emptyset$.?

Say $A = \emptyset$ and $B \neq \emptyset$. Then $B \neq A$.

But $A \times B = \emptyset$ and $B \times A = \emptyset$. So not true.

3 (c) Prove $A \times \emptyset = \emptyset$

Suppose $(a,b) \in A \times \emptyset$. This means $b \in \emptyset$. But this is impossible so $A \times \emptyset \subseteq \emptyset$ and $A \times \emptyset = \emptyset$.

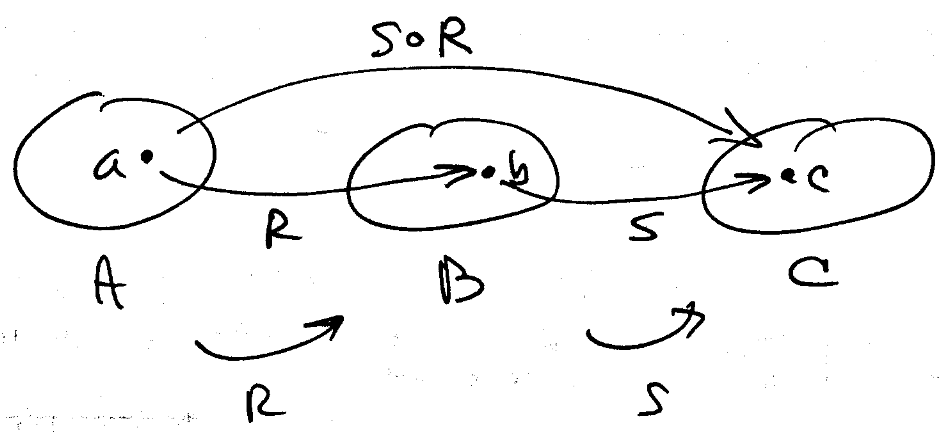
3.1 1(b) $A = \{1, 2, \{1,2\}\}$ $B = \{q, \{t\}, \pi\}$

$$A \times B = \{(1,q), (1,\{t\}), (1,\pi), (2,q), (2,\{t\}), (2,\pi), (\{1,2\},q), (\{1,2\},\{t\}), (\{1,2\},\pi)\}.$$

13 (a) [illegible]: R relation from A to B S reln from B to C.

$S \circ R = \{(a,c) \in A \times C : \exists\, b \in B$ such that $a R b$ and $b S c.\}$

$S \circ R$ relation from A to C



$\mathrm{Dom}(S \circ R) = \{a \in A : (\exists\, c \in C)\ (a,c) \in S \circ R\}$

$(\subseteq A)$ $\mathrm{Dom}(R) = \{a \in A : (\exists\, b \in B)\ (a,b) \in R\}.$

Pf: Let $a \in \mathrm{Dom}(S \circ R)$. Then there is a $c \in C$ such that $(a,c) \in S \circ R$. This means there is a $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. But this means ~~(a,b)∈f~~ $a \in \mathrm{Dom}(R)$. Hence $\mathrm{Dom}(S \circ R) \subseteq \mathrm{Dom}(R)$.

Equivalence Relations (cont'd)

Idea: Generalization of idea of "equality" or "equivalence".

② Equivalence relation.

<u>Def</u>: Let R be a relation on A.

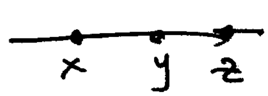
(i) R is <u>reflexive</u> if for all $x \in A$, xRx.

(ii) R is <u>symmetric</u> if for all $x, y \in A$, xRy implies yRx.

(iii) R is <u>transitive</u> if for all $x, y, z \in A$, xRy and yRz implies xRz.

<u>Examples</u>: Consider $\leq$ on $\mathbb{R}$, i.e. xRy means $x \leq y$. Reflexive? Means $\forall x \in \mathbb{R}$, $x \leq x$. So ~~TRUE~~ YES

Symmetric? Means $\forall x, y \in \mathbb{R}$ $x \leq y$ implies $y \leq x$. <u>NO.</u> eg. $1 \leq 2$ but $2 \nleq 1$. True only when $x = y$.

Transitive? Means $\forall x, y, z \in \mathbb{R}$, $x \leq y$ and $y \leq z$ implies $x \leq z$. 

. <u>YES</u>

Example: Consider the relation "divides" on $\mathbb{N}$. xRy means $x|y$. Reflexive? Always $x|x$ so <u>YES</u>.

symmetric? $x|y \Rightarrow y|x$? $1|2$ but $2 \nmid 1$ <u>NO</u>

Transitive? $x|y$ and $y|z$ implies $x|z$? <u>YES</u>

$x|y$ means $mx = y$ & $y|z$ means $ny = z$ so $mnx = ny = z$

Suppose change R to xRy iff $\underline{x|y \text{ or } y|x}$.
This makes R symmetric. Reflexive? YES.
Transitive? $(x|y \text{ or } y|x)$ and $(y|z \text{ or } z|y)$
$\Longrightarrow (x|z \text{ or } z|x)$. NO

$x=6 \quad y=3 \quad z=9$

$6R3$? Yes $\quad 3R9$? YES $\quad 6R9$? NO

Example : Define R on $\mathbb{R}$ by xRy iff $xy>0$.
Reflexive? NO. $0 \not R 0$
Symmetric? $xy>0 \Rightarrow yx>0$. YES.
Transitive? $xy>0$ and $yz>0 \Rightarrow xz>0$ YES.

What if I say xRy iff $xy \geq 0$.
Reflexive? YES.
Symmetric? YES.
Transitive? NO. $\quad 3R0$ and $0R-3$
but $3 \not R -3$.

Def: A relation R on A is an equivalence relation if it is reflexive, symmetric and transitive.

Example: Relation is "mod N" on $\mathbb{Z}$.

$x R y$ iff $N \mid (x-y)$

Reflexive? $x R x$ is $N \mid (x-x)$ or $N \mid 0$. True.
so YES

symmetric? $N \mid (x-y) \Rightarrow N \mid (y-x)$. True.
$N \mid (x-y)$ means $\exists\, n \in \mathbb{Z}$ s.t. $nN = x-y$. But
$(-n)N = (y-x)$ so $N \mid (y-x)$.

Transitive? If $N \mid (x-y)$ and $N \mid (y-z)$ then $N \mid (x-z)$

Pf: If $N \mid (x-y)$ and $N \mid (y-z)$ then $N \mid ((x-y)+(y-z))$
or $N \mid (x-z)$.

Example: Define $\square$ on $\mathbb{R}$ by $x \,\square\, y$ iff $|x| = |y|$

Reflexive? $x \,\square\, x$ means $|x| = |x|$. YES.

Symmetric? $x \,\square\, y \Rightarrow y \,\square\, x$? YES

Transitive? $x \,\square\, y$ and $y \,\square\, z \Rightarrow x \,\square\, z$?

$|x| = |y|$ and $|y| = |z| \Rightarrow |x| = |z|$ YES

(3) Equivalence class.

<u>Def</u>: Let R be an equiv. relation on A and let $x \in A$. The equivalence <u>class</u> of x, denoted x/R and read "x mod R" is the set

$$x/R = \{ y \in A : xRy \}$$

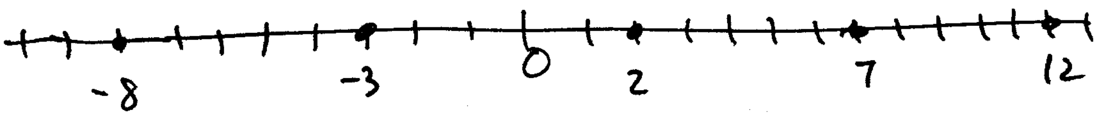
<u>Example</u>: (i) Say R is "mod 5" on $\mathbb{Z}$.

What is $7/R$? $\{2, 7, 12, 17, -3, 87, -142, \ldots\}$

$\boxed{xRy \text{ iff } 5 \mid (x-y)}$

$= \{5n+2 : n \in \mathbb{Z}\}$.

$= \{5n+7 : n \in \mathbb{Z}\}$.



Note that $7/R = 2/R = 17/R$ etc...

(ii) Say R is xRy iff $|x| = |y|$ (R is on $\mathbb{R}$).

$3/R = \{3, -3\}$.

eg #4 (f). $X = \{m, n, p, q, r, s\}$ $\mathcal{P}(X)$ has $2^6 = 64$ elts.

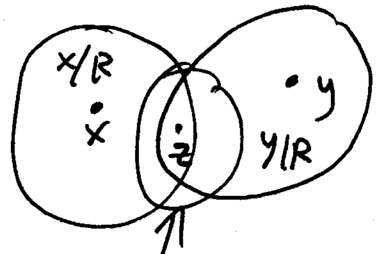
R on $\mathcal{P}(X)$ defined by $A R B$ iff $\overline{A} = \overline{B}$.

$\{m\}/R = \{\{m\}, \{n\}, \{p\}, \{q\}, \{r\}, \{s\}\}$

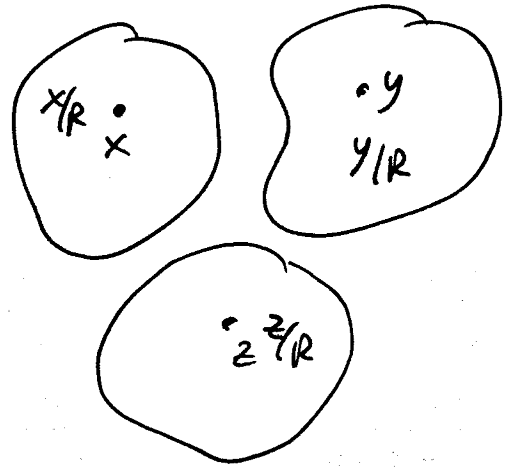
3.3 Partitions

① Partitions:

Spse R is an equiv relation on



can't happen.



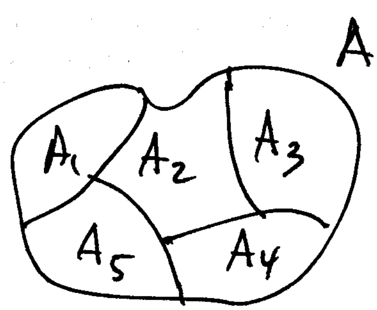
Def: Given $A \neq \emptyset$, a partition of A is a collection of subsets of A, (denoted by $\mathcal{A}$) such that

(i) $\emptyset \in \mathcal{A}$

(ii) If $X, Y \in \mathcal{A}$ and $X \neq Y$, then $X \cap Y = \emptyset$
($\mathcal{A}$ is a pairwise disjoint collection)

(iii) $\bigcup_{X \in \mathcal{A}} X = A$.

Example:



__Example__: Equiv classes of $\equiv_m$ (equivalence mod m) ($x \equiv_m y$ iff $m \mid (x-y)$). on $\mathbb{Z}$.

Say $m = 5$.

$$0/R = \{\dots, -10, -5, 0, 5, 10, 15, 20, \dots\}$$
$$= \{5n : n \in \mathbb{Z}\}$$

$$1/R = \{\dots, -14, -9, -4, 1, 6, 11, 16, \dots\}$$
$$= \{5n+1 : n \in \mathbb{Z}\}.$$

$$2/R = \{5n+2 : n \in \mathbb{Z}\}$$
$$= \{\dots, -8, -3, 2, 7, 12, 17, \dots\}$$

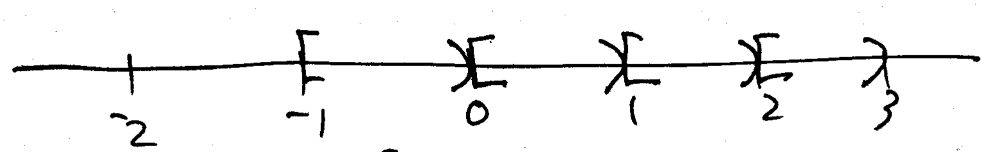
$$3/R = \{5n+3 : n \in \mathbb{Z}\}$$

$$4/R = \{5n+4 : n \in \mathbb{Z}\}$$

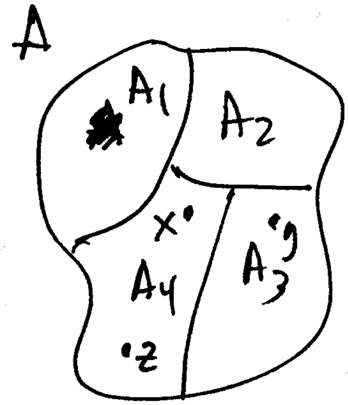
This is a partition of $\mathbb{Z}$.

__Example__: $\mathbb{Z} = \{\text{even integers}\} \cup \{\text{odd integers}\}$

__Example__: $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [n, n+1)$



② Remark: Every partition of a set A defines a relation on A. How?



Define R by xRy iff x and y are in same element of the partition.

More specifically: Given a partition $\mathcal{Q}$ of a set A, define R on A by: xRy iff there exists $B \in \mathcal{Q}$ such that $x, y \in B$.

Example: Define the partition $\{A_0, A_1, A_2, A_3, A_4\}$ of $\mathbb{Z}$ by

$$A_i = \{5n+i : n \in \mathbb{Z}\}$$

What is the relation? xRy iff for some $i = 0,1,2,3,$ or 4, $x, y \in A_i$. That is,

$x = 5n+i$ and $y = 5k+i$ some $n, k \in \mathbb{Z}$

This means $x - y = 5(n-k)$, i.e. $5 \mid (x-y)$

Examply: $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} [n, n+1)$

What is the relation?

$3.5 \, R \, 3$? Yes $\qquad -9.3 \, R \, -9$? No

$2.9 \, R \, 3.1$? No.

Say $x R y$ iff $[\![x]\!] = [\![y]\!]$

$\left[[\![x]\!] = \right.$ greatest integer $\left. \leq x \right]$

③ Does every relation define a partition?
No. Only equivalence relations.

Thm: Let R be an equivalence relation on a non-empty set A. Then the equivalence classes of R form a partition of A.

Pf: Let $\mathcal{B} = \{x/R : x \in A\}$. Must show that

(i) If $x/R \in \mathcal{B}$ then $x/R \neq \emptyset$.

(ii) If $x/R \cap y/R \neq \emptyset$ then $x/R = y/R$.

(iii) $\bigcup_{x\in A} x/R = A$.

To see (i) note that for each $x \in A$, $x \in x/R$ since R is reflexive. Hence $x/R \neq \emptyset$.

To see (iii) note that since $x/R \subseteq A$ for each $x \in A$ then $\bigcup_{x \in A} x/R \subseteq A$ and ~~for~~ if $x \in A$ then $x \in x/R$ so $x \in \bigcup_{x \in A} x/R$, so $A \subseteq \bigcup_{x \in A} x/R$.

Hence $\bigcup_{x \in A} x/R = A$.

To see (ii) suppose that $z \in x/R \cap y/R$. Then xRz and yRz. By symmetry, zRy and by transitivity xRy. This means that $y \in x/R$ and since yRx, $x \in y/R$. We still need to show that $x/R = y/R$. Let $w \in x/R$. Then xRw and since R is symmetric wRx. Since ~~xe~~ $y \in x/R$, xRy so wRy and ~~yRw~~ yRw. Hence $w \in y/R$, and so $x/R \subseteq y/R$. Let $w \in y/R$. Then yRw and wRy. Since $x \in y/R$, yRx so wRx and xRw. Hence $w \in x/R$, so $y/R \subseteq x/R$. Therefore $x/R = y/R$.

Thm: Let $\mathcal{B}$ be a partition of a non-empty set A. Define the relation Q on A by xQy iff there is a $C \in \mathcal{B}$ such that $x, y \in C$. Then

(i) Q is an equivalence relation and

(ii) $\mathcal{B} = \{x/R : x \in A\}$.