② Combinations and Permutations

Thm 2.20

e.g. Find the number of possible outcomes of n coin tosses (i.e. either heads or tails)

$$\underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \cdots\ \underline{2} \leftarrow \text{\# outcomes of each}$$

n tosses

total # of outcomes is 2^n.

Def: A permutation of a set A with n elements is an arrangement (or ordering) of the elements of A.

e.g $A = \{a, b, c, d\}$

a b c d
b a c d
c a b d
d a c b
etc.

Thm: There are $n!$ permutations of a set A with $\overline{A} = n$.

Why? n choices for first
$n-1$ " for second
$n-2$ " " third
$\vdots$

so $n(n-1)(n-2)\cdots(2)(1)$ choices $= n!$

Thm: The number of ~~per~~ permutations of r objects taken from a set with n objects $(r \leq n)$ is

$$\frac{n!}{(n-r)!}$$

Tasks: T_1 : choose first $\qquad n$

T_2 : choose second $\qquad n-1$

$\qquad\qquad\qquad n-2$

$\vdots \qquad \vdots \qquad \vdots$

T_r : choose rth $\qquad n-r+1$

So total # of such permutations is:

$$\frac{n(n-1)(n-2)\cdots(n-r+1)\cancel{(n-r)}\cancel{(n-r-1)}\cdots(2)(1)}{(n-r)\cancel{(n-r-1)}\cdots(2)(1)}$$

$$= \frac{n!}{(n-r)!}$$

Def: A combination of n element taken r at a time is an r-element subset of an n-element set.

Thm: The number of combinations of n elements taken r at a time is

$$\frac{n!}{(n-r)!\,r!} = \binom{n}{r}$$ ← binomial coefficient ← "n choose r"

e.g. $A = \{a, b, c, d\}$

Permutations of 3 objects:

abc	bac	cab
abd	bad	cba
acd	bca	etc.
acb	bda	⋮
adb	bcd	⋮
adc	bdc	

total of $4 \cdot 3 \cdot 2 = 24$

Combinations of 3 objects:

total of: $\frac{4 \cdot 3 \cdot 2}{3 \cdot 2} = 4$

eg #4(c)

Team 1	Team 2	Team 3	Team 4
4	3	3	3

Total of 13 pitchers, 5 LHP, 8 RHP

#ways to select 4 pitchers incl 2 LHP $= \binom{5}{2}\binom{8}{2} = \frac{5!}{3!\,2!} \cdot \frac{8!}{6!\,2!}$

$= \frac{5 \cdot 4}{2} \cdot \frac{8 \cdot 7}{2} = 280$

③ Binomial Theorem

Thm: $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$

e.g. $\sum_{r=0}^{n} \binom{n}{r} = 2^n$

Pf: Let $a=b=1$

$2^n = (a+b)^n = (1+1)^n = \sum_{r=0}^{n} \binom{n}{r} 1^r 1^{n-r} = \sum_{r=0}^{n} \binom{n}{r}$

Thm 2.24

(d) $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$

$A = \{x_1, x_2, \ldots, x_n\}$

$= \{x_1, x_2, \ldots, x_{n-1}\} \cup \{x_n\}$

subsets of A with r elements excluding x_n $= \binom{n-1}{r}$

subsets of A with r elements including x_n $= \binom{n-1}{r-1}$

3.1 Cartesian Products and Relations

① Cartesian products

Def: Let A and B be sets. The Cartesian (or cross) product of A and B, denoted $A \times B$ is the set

$$A \times B = \{(a,b) \text{ such that } a \in A \text{ and } b \in B\}$$

– an element (a,b) of $A \times B$ is an ordered pair, i.e. a pair where the order is important, i.e. $(a,b) \neq (b,a)$

Example: $A = \{1,2\}$ $B = \{2,3,4\}$

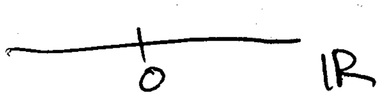
$$A \times B = \{(1,2), (1,3), (1,4), (2,2), (2,3), (2,4)\}$$

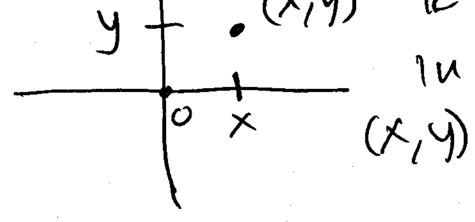
Note $\overline{\overline{A}} = 2$ $\overline{\overline{B}} = 3$ $\overline{\overline{A \times B}} = 6$

Thm: If $\overline{\overline{A}} = m$ and $\overline{\overline{B}} = n$ then $\overline{\overline{A \times B}} = mn$

– We say two elements $(a_1, b_1), (a_2, b_2) \in A \times B$ are equal iff $a_1 = a_2$ and $b_1 = b_2$.

Example: the product $\mathbb{R} \times \mathbb{R}$ (denoted usually by $\mathbb{R}^2$) is the (Cartesian) plane





in general $(x,y) \neq (y,x)$

Thm 3.1 : $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Pf: Let $(x,y) \in A \times (B \cap C)$. Then $x \in A$ and $y \in B \cap C$. But this means $y \in B$ and $y \in C$. Therefore $(x,y) \in A \times B$ and $(x,y) \in A \times C$. Hence $(x,y) \in (A \times B) \cap (A \times C)$.

Let $(x,y) \in (A \times B) \cap (A \times C)$. This means $(x,y) \in A \times B$ and $(x,y) \in A \times C$. Therefore $x \in A$, $y \in B$ and $y \in C$. Hence $y \in B \cap C$ and since $x \in A$, $(x,y) \in A \times (B \cap C)$.

2 Relations

Def: Let A, B be sets. A relation from A to B, denoted R, is a subset of $A \times B$. We write aRb if $(a,b) \in R$ and $a \not R b$ if $(a,b) \notin R$. If $A = B$ we say R is a relation on A.

- think of R as relating each element of a subset of A to each element of a subset of B

<u>Example</u> Telephone faceplate

$\Delta = \{0,1,2,\ldots,9\}$ $\quad T' = \{A,B,C,\ldots,X,Y,Z\}$

R (from Δ to T') is given by

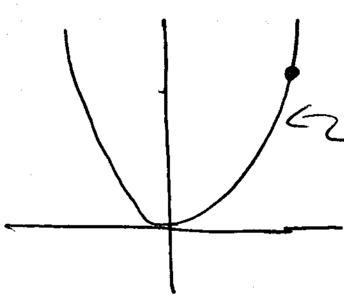
$R = \{(2,A),(2,B),(2,C),(3,D),(3,E),(3,F),\ldots\}$

— Note that 0, 1 ~~are~~ do not appear neither do Q or Z.

<u>Example</u>: Define the relation R on $\mathbb{R}$ by

$R = \{(x,y) : y = x^2\}$ $\quad (3,-9) \notin R$ $\quad x \; y$ $\quad (-3,9) \in R$ $\quad (3,9) \in R$ $\quad (2,-1) \notin R$

Every $x \in \mathbb{R}$ appears as first component in a pair in R but only $y \geq 0$ appear.



this curve <u>is</u> R.

<u>Def</u>: Let R be a relation from A to B then

domain $\to$ $\mathrm{Dom}(R) = \{x \in A : \text{there exists } y \in B \text{ such that } (x,y) \in R\}$.

range $\to$ $\mathrm{Rng}(R) = \{y \in B : \text{there exists } x \in A \text{ such that } (x,y) \in R\}$.

Example (Telephone)

$\mathrm{Dom}(R) = \{2, 3, \ldots, 9\}$

$\mathrm{Rng}(R) = T - \{Q, Z\}$.

Example: $R = \{(x, y) : y = x^2\}$

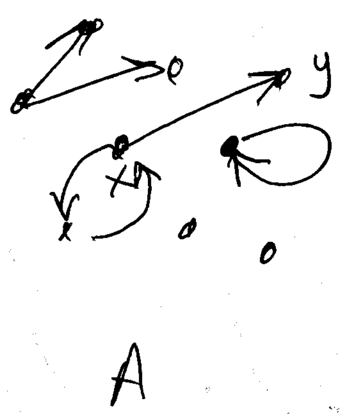
$\mathrm{Dom}(R) = \mathbb{R}$

$\mathrm{Rng}(R) = \{y \in \mathbb{R} : y \geq 0\} = [0, \infty)$

(3) Graphs

Given a set A and a relation R on A we can represent R by a directed graph as follows:

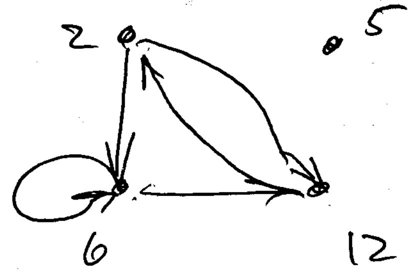
(i) think of A as a set of vertices



(ii) think of R as giving "arrows" that connect points of A.

i.e. draw an arrow from x to y iff xRy

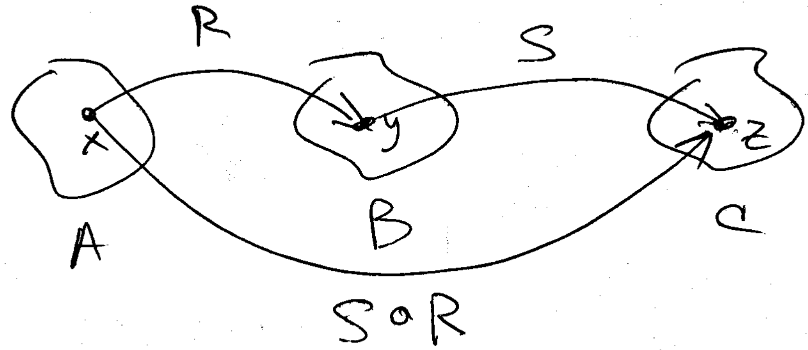
e.g. Fig 3.6



$A = \{2, 5, 6, 12\}$

$R = \{(2,6), (2,12), (6,6), (6,12), (12,2)\}$

Composite relations



Given relations R from A to B and S from B to C we define the relation $S \circ R$ from A to C by

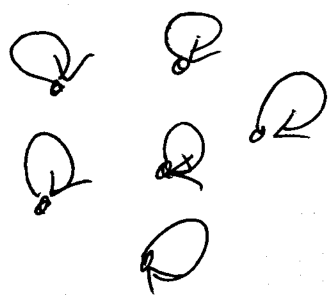
$(x, z) \in S \circ R$ iff there exists $y \in B$ such that $(x, y) \in R$ and $(y, z) \in S$.

#9 (e) $S \circ R = \{(1,5), (2,4), (5,4)\}$

3.2 Equivalence Relations

① Motivation

We know that "equality" is a relation on any set A, i.e. $x R y$ iff $x = y$.



Loosen our thinking a little.
Might want to think of 2 elements in a set A as "equivalent" even if they are not identical.

e.g. modular arithmetic.

Let $N \in \mathbb{N}$, $x, y \in \mathbb{Z}$

We say $x \equiv y \mod N$ if $N \mid (x-y)$

$7 \equiv 2 \mod 5 \qquad 83 = 13 \mod 7$

The relation here would be on $\mathbb{Z}$ and $x R y$ iff $N \mid (x-y)$

e.g. Calculus

$$\int 2x\,dx = x^2 + \underline{c}$$

Look at f(x) and g(x) as equivalent if $f'(x) = g'(x)$

So relation on all continuous functions on $\mathbb{R}$ would be $f \, R \, g$ if $f' = g'$.

Q: What properties might R have to have to make it useful as a notion of "equivalence"?