Exam 3 2.1–2.4

2.2 10(a) 11(a)(c) 15(a)

2.1 16, 13

2.3 10(b), 17(a)

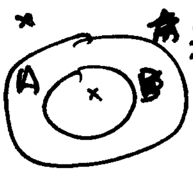
16) Prove $X = Y$.

Pf: (1) Show $X \subseteq Y$. Let $x \in X$. Then $x \in \mathbb{Z}$ and $|x| \le 3$. Then $x = -3, -2, -1, 0, 1, 2,$ or 3. Therefore $x \in Y$.

(2) Show $Y \subseteq X$. Let $y \in Y$. Then $y = -3, -2, -1, 0, 1, 2$ or 3. In each case $y \in \mathbb{Z}$ and $|y| \le 3$. Therefore $y \in X$.

Therefore $X = Y$.

2.2. 10(a) Prove $A \subseteq B$ if and only if $A - B = \emptyset$



Pf: $(\Rightarrow)$ (That is, prove if $A \subseteq B$ then $A - B = \emptyset$)
Assume $A \subseteq B$, and that $x \in A - B$. This means that $\underline{x \in A}$ and $x \notin B$, or that $x \in \tilde{B}$. But $A \subseteq B$ means $\tilde{B} \subseteq \tilde{A}$ so $x \in \tilde{B}$ means $\underline{x \in \tilde{A}}$. But this contradicts $x \in A$. So $A - B = \emptyset$.

$(\Leftarrow)$ (that is, prove if $A - B = \emptyset$ then $A \subseteq B$)
Assume $A - B = \emptyset$, and let $x \in A$. Then either $x \in B$ or $x \in \tilde{B}$. If $x \in \tilde{B}$ then $x \in A \cap \tilde{B} = A - B$. But this is impossible since $A - B = \emptyset$. Therefore $x \in B$. Hence $A \subseteq B$.

Let $x \in A - B$. Then $x \in A \cap \tilde{B}$ so $x \in A$. But $x \in \emptyset$ so $x \in B$ and $x \in \tilde{B}$. ~~Since~~ Therefore $x \in B$. Hence $A - B \subseteq B$. not complete.

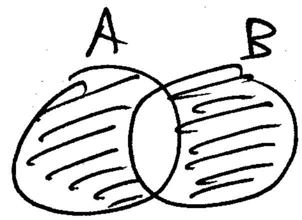
11 (a) Prove: If $C \subseteq A$ and $D \subseteq B$ then $C \cap D \subseteq A \wedge B$.

Pf: Assume $C \subseteq A$ and $D \subseteq B$, and let $x \in C \wedge D$. Then $x \in C$ and since $C \subseteq A$, $x \in A$. Also $x \in D$ and since $D \subseteq B$, $x \in B$. Therefore $x \in A \wedge B$.

(c) Prove: If $C \subseteq A$, $D \subseteq B$, and $A \cap B = \emptyset$ then $C \cap D = \emptyset$.

Pf: Need to show $C \cap D = \emptyset$. Let $x \in C \cap D$. Then $x \in C$ and since $C \subseteq A$, $x \in A$. Also $x \in D$ and since $D \subseteq B$, $x \in B$. Therefore $x \in A \cap B = \emptyset$. Hence $C \cap D \subseteq \emptyset$ so $C \cap D = \emptyset$.

15 (a) $A \triangle B = (A-B) \cup (B-A)$



Prove $A \triangle B = B \triangle A$

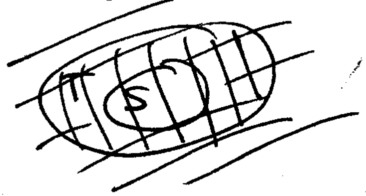
Pf: $A \triangle B = (A-B) \cup (B-A) = (B-A) \cup (A-B) = B \triangle A$.

[$A-B \neq B-A$]

$S = (A-B) \cup (B-A)$

$T = (B-A) \cup (A-B)$

$S \subseteq T$ $\qquad$ $\tilde{S} \cup T = \mathcal{U}$ $\qquad$ [$P \Rightarrow Q$ $\quad \sim P \vee Q$]



2.3 10(b) $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ $\Gamma \subseteq \Delta$

Show $\bigcap_{\alpha \in \Delta} A_\alpha \subseteq \bigcap_{\alpha \in \Gamma} A_\alpha$

Pf

Let $x \in \bigcap_{\alpha \in \Delta} A_\alpha$. Then for all $\alpha \in \Delta$, $x \in A_\alpha$.

But since $\Gamma \subseteq \Delta$, $x \in A_\alpha$ for all $\alpha \in \Gamma$. Therefore

$x \in \bigcap_{\alpha \in \Gamma} A_\alpha$.

17(a) $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$. Let $k, \cancel{m} \in \mathbb{N}$, $\cancel{k \leq m}$

Prove $\bigcup_{i=1}^{k+1} A_i = \bigcup_{i=1}^{k} A_i \cup A_{k+1}$

Pf: (1) Let $x \in \bigcup_{i=1}^{k+1} A_i$. Then for some $i = 1, 2, \ldots, k+1$, $x \in A_i$. Clearly $i \in \{1, 2, \ldots, k\}$ or $i = k+1$. In the first case, $x \in \bigcup_{i=1}^{k} A_i$ and in second case $x \in A_{k+1}$

Hence $x \in \bigcup_{i=1}^{k} A_i \cup A_{k+1}$.

(2) Let $x \in \bigcup_{i=1}^{k} A_i \cup A_{k+1}$. Then $x \in \bigcup_{1}^{k} A_i$ or $x \in A_{k+1}$.

In either case, $x \in \bigcup_{i=1}^{k+1} A_i$.

<u>Recap</u>: Induction: If $S \subseteq \mathbb{N}$ satisfies (i) $1 \in S$ and (ii) for all $n \in \mathbb{N}$, if $n \in S$ then $n+1 \in S$, then $S = \mathbb{N}$.

<u>Example</u>: Prove that for all $n \in \mathbb{N}$, if A is a set with n elements, then $\mathcal{P}(A)$ has 2^n elements.

<u>Pf</u>: (1) If $n=1$, then we can write $A = \{x_1\}$ then $\mathcal{P}(A) = \{\emptyset, \{x_1\}\}$ so $\mathcal{P}(A)$ has $2^1 = 2$ elements. So result holds for $n=1$.

(2) Assume <u>the result</u> holds for n, and let A be a set with $n+1$ elements. We can enumerate A by $A = \{x_1, x_2, \dots, x_n, x_{n+1}\}$. Let $B = \{x_1, \dots, x_n\}$. Every subset of A either contains x_{n+1} or it does not. If it does not then it is a subset of B. By induction hypothesis there are 2^n such subsets of A. If it does, then it can be written as $A' \cup \{x_{n+1}\}$ where $A' \subseteq B$. There are 2^n such subsets of A. So all together there are $2^n + 2^n = 2^{n+1}$ subsets of A. In other words $\mathcal{P}(A)$ has 2^{n+1} elements.

Example: Show that for all $n \in \mathbb{N}$, and any $r \in \mathbb{R}$, $r \neq 1$

$$\sum_{i=0}^{n-1} r^i = \frac{r^n - 1}{r-1}$$

Pf: Informal argument

$$\sum_{i=0}^{n-1} r^i = 1 + r + r^2 + r^3 + \cdots + r^{n-1}$$

$$r\left(\sum_{i=0}^{n-1} r^i\right) = r\left(1 + r + \cdots + r^{n-1}\right)$$

$$= r + r^2 + r^3 + \cdots + r^{n-1} + r^n$$

$$\left(\sum_{i=0}^{n-1} r^i\right) - r\left(\sum_{i=0}^{n-1} r^i\right) = (1-r)\left(\sum_{i=0}^{n-1} r^i\right)$$

$$= 1 + \cancel{r} + \cancel{r^2} + \cdots + \cancel{r^{n-1}}$$

$$- \left(\cancel{r} + \cancel{r^2} + \cdots + \cancel{r^{n-1}} + r^n\right) = 1 - r^n$$

$$\therefore \sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r} = \frac{r^n - 1}{r-1}.$$

Formal proof: (1) If $n = 1$ the result says that

$\sum_{i=0}^{0} r^i = 1 = \frac{r-1}{r-1}$ which is true.

(2) Now assume that $\sum_{i=0}^{n-1} r^i = \frac{r^n - 1}{r-1}$. Then

$$\sum_{i=0}^{n} r^i = \left(\sum_{i=0}^{n-1} r^i\right) + r^n = \frac{r^n - 1}{r-1} + r^n = \frac{\cancel{r^n} - 1 + r^{n+1} - \cancel{r^n}}{r-1}$$

$$= \frac{r^{n+1} - 1}{r-1}.$$

③ Generalized PMI.

Let $k \in \mathbb{N}$. If $S \subseteq \mathbb{N}$ satisfies

(i) $k \in S$

(ii) for all $n \in \mathbb{N}$, $n \geq k$, if $n \in S$ then $n+1 \in S$

then $\{k, k+1, \dots\} \subseteq S$.

- This is induction starting at k instead of at 1.

e.g. #9 (c) Prove that $n! > 2^{n+2}$ for $n \geq 6$.

n	$n!$	2^{n+2}
1	1	8
2	2	16
3	6	32
4	24	64
5	120	128
6	720	256
7	5040	512
8	40320	1024

Pf: (1) If $n=6$ then $n! = 720$ and $2^{n+2} = 256$ so $n! > 2^{n+2}$ in this case.

(2) Assume that for some $n \geq 6$, $n! > 2^{n+2}$. Will show that $(n+1)! > 2^{n+3}$. Since $n+1 > 2$ and since $n! > 2^{n+2}$, $(n+1)n! > 2 \cdot 2^{n+2}$. But this is the same as $(n+1)! > 2^{n+3}$.

$$(n+1)! = (n+1)\, n!$$
$$2^{n+3} = 2 \cdot 2^{n+2}$$

2.5 Equivalent forms of Induction.

① Principle of Complete Induction (PCI)

If $S \subseteq \mathbb{N}$ satisfies

(i) for all $n \in \mathbb{N}$, if $\{1, 2, \dots, n-1\} \subseteq S$ then $n \in S$.

then $S = \mathbb{N}$.

— Why no base case? Really there just hidden.

If $n = 1$ then $\{1, 2, \dots, n-1\} = \emptyset$ so (i) means $\underline{\emptyset \subseteq S} \Rightarrow \underline{1 \in S}$.

— in a real proof, you would prove the base case separately as usual.

— Compare with PMI, we had "$n \in S \Rightarrow n+1 \in S$"

Here we have stronger hypothesis, namely $\{1, \dots, n\} \in S \Rightarrow n+1 \in S$.

Example: Prove that every integer $n \geq 2$ is either prime or a product of primes.

Pf: (1) Let $n = 2$. Then n is prime so result holds in this case.

(2) Suppose result holds for the set $\{2, \dots, n\}$ for some $n \in \mathbb{N}$. Want to show it for $n+1$.

If $n+1$ is prime we are done, If not then it is composite, i.e. $n+1 = ab$ for some $2 \le a < n+1$ and ~~b~~ $2 \le b < n+1$. By induction hypothesis, a and b are either prime or products of primes. Hence $n+1$ is a product of primes.

[To show something like $P \vee Q$ enough to show if $\sim P$ then Q (since $P \vee Q$ equiv. to $\sim P \Rightarrow Q$)]

Example: Fibonacci sequence.

$f_1 = 1$, $f_2 = 1$, $f_3 = 1+1 = 2$, $f_4 = 1+2 = 3$, $f_5 = 3+2 = 5$

$f_6 = 8$, $f_7 = 13$, etc. In general $f_n = f_{n-1} + f_{n-2}$.

~~Prove that~~

Let $\alpha > \beta$ be the roots of $x^2 = x+1$.

Then for all $n \in \mathbb{N}$, $$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Pf: (1) If $n=1$, $f_1 = \frac{\alpha-\beta}{\alpha-\beta} = 1$. If $n=2$ then

$$f_2 = \frac{\alpha^2-\beta^2}{\alpha-\beta} = \frac{(\alpha+\beta)(\alpha-\beta)}{\alpha-\beta} = \alpha+\beta = 1.$$

$$\boxed{\begin{aligned} &x^2 - x - 1 = 0 \\ &(x-\alpha)(x-\beta) = 0 \\ &x^2 - (\alpha+\beta)x + \alpha\beta = 0 \end{aligned}}$$

So result holds for $n = 1, 2$.

(2) Now suppose result holds for $\{1, 2, \ldots, n-1\}$ for some $n > 2$. Will show it holds for n.

$$f_n = f_{n-1} + f_{n-2}$$

$$= \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + \frac{\alpha^{n-2} - \beta^{n-2}}{\alpha - \beta}$$

$$= \frac{\alpha^{n-2}(\alpha + 1) - \beta^{n-2}(\beta + 1)}{\alpha - \beta}$$

$$= \frac{\alpha^{n-2}(\alpha^2) - \beta^{n-2}(\beta^2)}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Hence result holds for all $n \in \mathbb{N}$. $\square$

<u>NB:</u> $x^2 = x + 1$

$x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} \qquad \alpha = \frac{1+\sqrt{5}}{2} \leftarrow \text{golden ratio} \approx 1.6$$

$$\beta = \frac{1-\sqrt{5}}{2}$$

Interesting: f_n always an integer

but $$f_n = \frac{\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]}{\sqrt{5}}$$

2-6 Principles of Counting.

① Notation and Basics

If A is a finite set then $\overline{\overline{A}}$ is the number of elements in A (usually $\#A$ or $|A|$)

e.g. $\overline{\overline{\{a,b,c\}}} = 3$

$\overline{\overline{\{\{a,b\},\{c,d\},\{a,b,c\}\}}} = 3$

$\overline{\overline{\{1,2,\dots,n\}}} = n$

Thm: ① If A and B are disjoint then

$$\overline{\overline{A \cup B}} = \overline{\overline{A}} + \overline{\overline{B}}$$

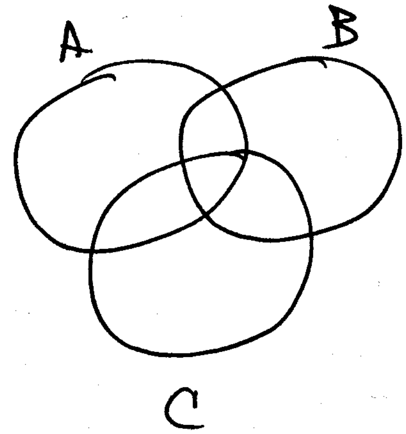
② If $\{A_i \cancel{*} : i = 1,2,\dots,n\}$ is a ~~d~~ disjoint indexed family then

$$\overline{\overline{\bigcup_{i=1}^{n} A_i}} = \sum_{i=1}^{n} \overline{\overline{A_i}}$$

③ If A and B are finite sets then

$$\overline{\overline{A \cup B}} = \overline{\overline{A}} + \overline{\overline{B}} - \overline{\overline{A \cap B}}$$

eg 2



$\overline{\overline{A}} = 24$

$\overline{\overline{B}} = 21$

$\overline{\overline{C}} = 23$

$\overline{\overline{A \cup B}} = 37 \qquad \overline{\overline{A \cap B}} =$

$\overline{\overline{A \cup C}} = \qquad \overline{\overline{A \cap C}} = 11$

$\overline{\overline{B \cup C}} = 33 \qquad \overline{\overline{B \cap C}} = 11$

$\overline{\overline{A \cup B \cup C}} =$

$\overline{\overline{A \cap B \cap C}} =$

$\overline{\overline{B - C}} = \overline{\overline{B}} - \overline{\overline{B \cap C}}$

$10 = 21 - \overline{\overline{B \cap C}}$

$\overline{\overline{C - B}} = \overline{\overline{C}} - \overline{\overline{C \cap B}}$

$12 = \overline{\overline{C}} - 11$

$\overline{\overline{B \cup C}} = \overline{\overline{B}} + \overline{\overline{C}} - \overline{\overline{B \cap C}}$

$33 = 21 + 23 - 11$

(2) Combinations / Permutations

Thm 2.20

- How many subsets of a set A with n elements are there?

  Suppose $A = \{x_1, x_2, \ldots, x_n\}$

  For each x_i decide "in" or "out".

  $$2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2 = 2^n$$