Remember: In order to prove that $A = B$ (A and B are sets), it is enough to show $A \subseteq B$ and $B \subseteq A$.

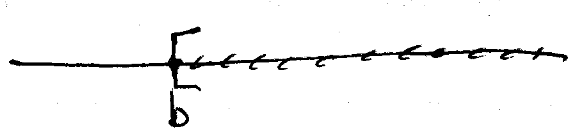
2.3 Extended set operations + indexed families

① Family of sets

$\mathcal{A}$ is a family (or collection) of sets if each element $A \in \mathcal{A}$ is itself a set.
[In short $\mathcal{A}$ is a set of sets].

Example: $\mathcal{A} = \{ \{1,2,3\}, \{3,4,5\}, \{6,7\} \}$
is a family of (finite) sets.
Note that $\{1,2,3\} \in \mathcal{A}$ but $\{1,2,3\} \not\subseteq \mathcal{A}$,
and also $1 \notin \mathcal{A}$, for example.

$\mathcal{B} = \{ [b, \infty) : b \in \mathbb{R} \}$ is a family of sets.



For each $b \in \mathbb{R}$, $[b, \infty) \in \mathcal{B}$, so $[1, \infty) \in \mathcal{B}$
$[-3, \infty) \in \mathcal{B}$, $[787, \infty) \in \mathcal{B}$, $[\sqrt{2}, \infty) \in \mathcal{B}$, etc...
There are infinitely many elements in $\mathcal{B}$.

② <u>Def</u>: Let $\mathcal{A}$ be a family of sets.

$$\bigcup_{A\in\mathcal{A}} A = \{x : x\in A \text{ for some } A\in\mathcal{A}\}$$

Think: $A\cup B\cup C$, $x\in A\cup B\cup C$
if ~~there is a~~ x is in <u>one</u> of the sets A, B or C

$$= \{x : (\exists A)((A\in\mathcal{A})\wedge(x\in A))\}$$

$$= \{x : (\exists A\in\mathcal{A})(x\in A)\}$$

<u>e.g.</u> $\mathcal{A} = \{\{1,2,3\},\{3,4,5\},\{6,7\}\}$

$$\bigcup_{A\in\mathcal{A}} A = \{1,2,3\}\cup\{3,4,5\}\cup\{6,7\}$$
$$= \{1,2,3,4,5,6,7\}.$$

<u>e.g.</u> $\mathcal{B} = \{[b,\infty) : b\in\mathbb{R}\}$

$$\bigcup_{B\in\mathcal{B}} B = (-\infty,\infty) = \mathbb{R}$$

Pf: Let $x \in \bigcup_{B \in \mathcal{B}} B$. Then for some $b \in \mathbb{R}$, $x \in [b, \infty)$. But $[b, \infty) \subseteq \mathbb{R}$, so $x \in \mathbb{R}$.

Therefore $\bigcup_{B \in \mathcal{B}} B \subseteq \mathbb{R}$.

Now let $x \in \mathbb{R}$. Need to show that $x \in [b, \infty)$ for some $b \in \mathbb{R}$. But clearly $x \in [x, \infty) \in \mathcal{B}$ so that $x \in \bigcup_{B \in \mathcal{B}} B$. Therefore $\mathbb{R} \subseteq \bigcup_{B \in \mathcal{B}} B$.

③ Def: Let $\mathcal{A}$ be a family of set.

$$\bigcap_{A \in \mathcal{A}} A = \{x : x \in A \text{ for all } A \in \mathcal{A}\}$$

$$= \{x : (\forall A)(A \in \mathcal{A} \Rightarrow x \in A)\}$$

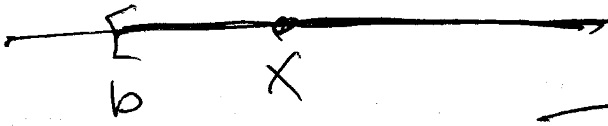
$$= \{x : (\forall A \in \mathcal{A})(x \in A)\}.$$

e.g. $\mathcal{A} = \{\{1,2,3\}, \{3,4,5\}, \{6,7\}\}$

$$\bigcap_{A \in \mathcal{A}} A = \{1,2,3\} \cap \{3,4,5\} \cap \{6,7\} = \emptyset$$

e.g. $\displaystyle\bigcap_{B \in \mathcal{B}} B = \emptyset$

$\mathcal{B} = \{[b,\infty) : b \in \mathbb{R}\}$

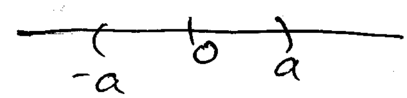


Pf: Let $x \in \displaystyle\bigcap_{B \in \mathcal{B}} B$. Then $x \in [b,\infty)$ for every $b \in \mathbb{R}$. This means that $b \leq x$ for every $b \in \mathbb{R}$, that is, that x is the largest real number. This contradicts the known fact that there is no such number.

OR Let $x \in \mathbb{R}$. Let $b = x+1$. Since $x+1 > x$, $x \notin [x+1,\infty)$. Hence $x \notin \displaystyle\bigcap_{B \in \mathcal{B}} B$. This means

$$\mathbb{R} \subseteq \widetilde{\left(\bigcap_{B \in \mathcal{B}} B\right)} \quad \text{so} \quad \bigcap_{B \in \mathcal{B}} B \subseteq \tilde{\mathbb{R}} = \emptyset.$$

e.g. Let $\mathcal{C} = \{(-a,a) : a \in \mathbb{R}\}$



Then $\displaystyle\bigcup_{C \in \mathcal{C}} C = \mathbb{R}$ $\quad \displaystyle\bigcap_{C \in \mathcal{C}} C = \{0\}$

③ Indexed Family of Sets.

e.g. $\mathcal{A} = \{\{1,2,3\}, \{3,4,5\}, \{6,7\}\}$

Let $A_1 = \{1,2,3\}$, $A_2 = \{3,4,5\}$, $A_3 = \{6,7\}$

Write $\mathcal{A} = \{A_1, A_2, A_3\}$

If we let $\Delta = \{1,2,3\}$, then the sets A in $\mathcal{A}$ are indexed by Δ.

Δ is arbitrary: Could say $\Delta = \{\alpha, \beta, \gamma\}$ and let $A_\alpha = \{1,2,3\}$, $A_\beta = \{3,4,5\}$, $A_\gamma = \{6,7\}$

Can write

$$\mathcal{A} = \{A_i : i \in \Delta\}.$$

e.g. $\mathcal{B} = \{[b,\infty) : b \in \mathbb{R}\}$. Can take $\Delta = \mathbb{R}$ then for each $b \in \Delta$, define $B_b = [b,\infty)$

Then $\mathcal{B} = \{B_b : b \in \Delta\}$.

e.g. (p 89) $\Delta = \{0,1,2,3,4\}$

$A_x = \{2x+4, 8, 12-2x\}$ for $x \in \Delta$.

$\mathcal{A} = \{A_x : x \in \Delta\} = \{A_0, A_1, A_2, A_3, A_4\}$

$$= \{ \{4,8,12\}, \{6,8,10\}, \underbrace{\{8,8,8\}}_{\{8\}}, \underbrace{\{10,8,6\}}_{\substack{\text{same as}\\ \{6,8,10\}\\ \text{already}\\ \text{here.}}}, \underbrace{\{12,8,4\}}_{\substack{\text{same as}\\ \{4,8,12\}\\ \text{already}\\ \text{here}}} \}$$

$$= \{ \{4,8,12\}, \{6,8,10\}, \{8\} \}.$$

④ Unions + Intersections

Let $\mathcal{A} = \{ A_i : i \in \Delta \}$

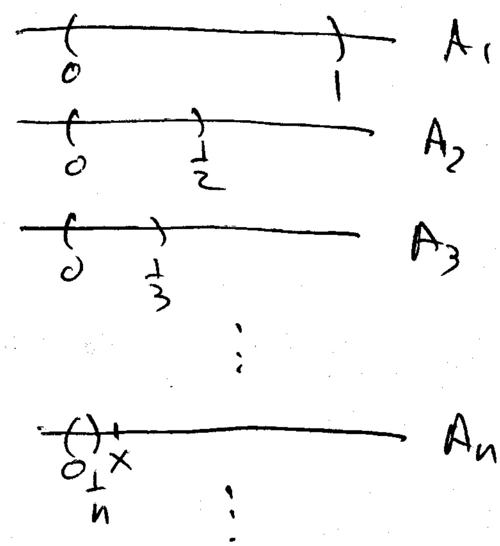
$$\bigcup_{i\in\Delta} A_i = \bigcup_{A\in\mathcal{A}} A = \{ x : x \in A_i \text{ for some } i \in \Delta \}$$

$$\bigcap_{i\in\Delta} A_i = \bigcap_{A\in\mathcal{A}} A = \{ x : x \in A_i \text{ for all } i \in \Delta \}$$

<u>eg</u> $A_n = (0, \frac{1}{n})$, $\mathcal{A} = \{ A_n : n \in \mathbb{N} \}$

$$\bigcup_{n\in\mathbb{N}} A_n = (0,1)$$

$$\bigcap_{n\in\mathbb{N}} A_n = \emptyset$$

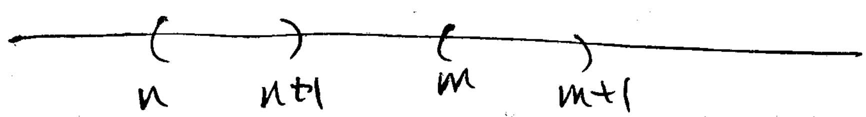


A family $\mathcal{A}$ of sets is pairwise disjoint if $A \cap B = \emptyset$ for every <u>distinct</u> $A, B \in \mathcal{A}$

An indexed family $\{A_\alpha : \alpha \in \Delta\}$ is pairwise disjoint if $A_\alpha \cap A_\beta = \emptyset$ ~~for every~~ whenever $A_\alpha \neq A_\beta$.

<u>eg</u> #1 (g) not pairwise disjoint

(m) $A_n = (n, n+1)$ for $n \in \mathbb{Z}$



A_n, A_m so pairwise disjoint.

2.4 Induction

① Induction gives a technique for proving statements of the form

$$(\forall n \in \mathbb{N})\ P(n)$$

where $P(n)$ is some open sentence defined on $\mathbb{N}$.

e.g. 1) For all $n \in \mathbb{N}$,

$$1+2+3+\cdots+n = \frac{n(n+1)}{2} \leftarrow P(n)$$

$n=1$: $1 = \frac{1\cdot 2}{2}$

$n=2$: $1+2 = \frac{2\cdot 3}{2}$

$n=4$: $\underbrace{1+2+3+4}_{10} = \underbrace{\frac{4(5)}{2}}_{10}$

2) For every set A with n elements, $\mathcal{P}(A)$ has 2^n elements. $\leftarrow P(n)$

3) $$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1} \leftarrow P(n)$$

② The Principle of Mathematical Induction (PMI) is a fundamental property of $\mathbb{N}$:

If $S \subseteq \mathbb{N}$ satisfies:

(i) $1 \in S$

(ii) for all $n \in \mathbb{N}$, if $n \in S$ then $n+1 \in S$

$[(\forall n \in \mathbb{N})(n \in S \Rightarrow n+1 \in S)]$

then $S = \mathbb{N}$.

How do you prove $(\forall n) P(n)$ using PMI?

Pf: Let $S = \{ n \in \mathbb{N} : P(n) \}$

(i) Show that $P(1)$ is true. (this is $1 \in S$)

(ii) Show that whenever $P(n)$ is true $P(n+1)$ is true $(P(n) \Rightarrow P(n+1))$. (this is $n \in S \Rightarrow n+1 \in S$)

(iii) Then $P(n)$ is true for all $n \in \mathbb{N}$. (this is $S = \mathbb{N}$)

e.g. Prove that for all $n \in \mathbb{N}$,

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

Pf: (i) Suppose that $n=1$. Then the result is $1 = \frac{1 \cdot 2}{2}$ which is true.

(ii) Suppose that $\underbrace{1+2+\cdots+n = \frac{n(n+1)}{2}}_{\text{induction hypothesis}}$. Want to show

$$1+2+3+\cdots+n+(n+1) = \frac{(n+1)(n+2)}{2}.$$

By the induction hypothesis

$$\begin{aligned}(1+2+3+\cdots+n)+(n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2}\end{aligned}$$

(iii) Therefore the result holds for all $n \in \mathbb{N}$.