EXAM 2 - Wednesday : Sections 1.5-1.7

1.6 6) (a) Say: $(\forall a,b,c,d \in \mathbb{R})\ \left((\overline{a+bi})+(\overline{c+di}) = \overline{(a+bi)+(c+di)}\right)$

Let $a,b,c,d \in \mathbb{R}$. Then $\overline{a+bi} = a-bi$ and $\overline{c+di} = c-di$. Therefore
$$(\overline{a+bi})+(\overline{c+di}) = a-bi+c-di$$
$$=(a+c)-(b+d)i.$$
~~But~~ And
$$\overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i} = (a+c)-(b+d)i$$
Therefore $(\overline{a+bi})+(\overline{c+di}) = \overline{(a+bi)+(c+di)}$.

7 (b) $(\exists M \in \mathbb{N})(\forall n)(n > M \Rightarrow \frac{1}{n} < 0.13)$

Notice $n > M \Rightarrow \frac{1}{n} < \boxed{\frac{1}{M} < 0.13}$

Pf: Take $M=10$. Then $\frac{1}{M} = .1 < .13$.
Therefore if $n \in \mathbb{N}$ with $n > M$ then
$$\frac{1}{n} < \frac{1}{M} = .1 < .13. \quad \blacksquare$$

1.5 6 (a) If $a|b$ then $a \le b$.

~~Prove~~ Prove A by contradiction
Assume $\sim A$ then derive $B \wedge \sim B$ $\cancel{B}$

$P \Rightarrow Q$ $\quad \sim(P \Rightarrow Q) \Leftrightarrow P \wedge \sim Q$

$P \wedge \sim Q$ is $a | b$ and $a > b$

$a | b$ means there is $k \in \mathbb{Z}$ such that $b = ka$. Since $a, b \in \mathbb{N}$, $k \in \mathbb{N}$

Hence $k \geq 1$ Therefore $a \leq ka = b$.

This contradicts assumption that $a > b$.

Assumed $P \wedge \boxed{\sim Q}$ then derived Q,

Better to do direct proof.

What about contrapositive? i.e. $\sim Q \Rightarrow \sim P$

$\sim Q$ is $a > b$ and $\sim P$ is $a \nmid b$.

Could say: if $a > b$ and $a, b \in \mathbb{N}$ ~~then there can be no $k \in \mathbb{N}$, $k \geq 1$ such that $ka = b$.~~ then for any $k \in \mathbb{N}$ $ka \geq a > b$. Therefore $ka \neq b$ for any $k \in \mathbb{N}$.

Pf (Direct) Suppose that $a | b$. Since $a, b \in \mathbb{N}$ there is a $k \in \mathbb{N}$ such that $ka = b$. But since $k \geq 1$, $a \leq ka = b$. Therefore $a \leq b$.

Pf (Contrapositive) Suppose that $a > b$. Then for every $k \in \mathbb{N}$, since $a, b \in \mathbb{N}$, $ka \geq a > b$. Therefore for all $k \in \mathbb{N}$, $ka \neq b$ so $a \nmid b$.

2.1 Basic concepts in Set Theory

① What is meant by a set.

- A set is a collection of objects with the property that membership in the collection is unambiguous.
- Often will write $A = \{x : P(x)\}$ where P is some open sentence so A is just the "truth set" of P

② Empty Set

- If $P(x)$ is false for all x then

$$\{x : P(x)\} = \emptyset$$

③ Subset

- $A \subseteq B$ (A is a subset of B) means every element of A is also an element of B
- Equivalently $(\forall x)(x \in A \Rightarrow x \in B)$

- If $A = \{x : P(x)\}$, $B = \{x : Q(x)\}$ then $A \subseteq B$ means $\forall x (P(x) \Rightarrow Q(x))$
- Often see $A \subset B$ not $A \subseteq B$

$A \subseteq B$ leaves open the possibility that $A = B$. If we want to exclude that we write $A \subsetneq B$ and say A is a <u>proper subset</u> of B.

- <u>Thm</u>: (a) $\emptyset \subseteq A$ for any set A

  (b) $A \subseteq A$ for any set A.

  <u>Pf</u>: (a) Need to show that $x \in \emptyset \Rightarrow x \in A$. But $x \in \emptyset$ is always false so $x \in \emptyset \Rightarrow x \in A$ is true for any A.

4 Equality of Sets.

- $A = B$ means $(\forall x)(x \in A \Leftrightarrow x \in B)$

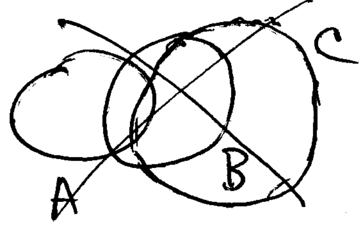
In other words $(\forall x)(x \in A \Rightarrow x \in B$ <u>and</u> $x \in B \Rightarrow x \in A)$.

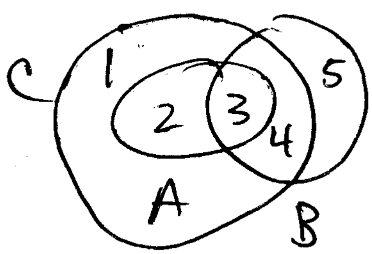
In still other words $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

- Thm: If A and B have no elements then $A = B$. (That is, the empty set is unique.)

  Pf: Need to show $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$. But both are true since both antecedents are false.

e.g. #5 (c) $A \not\subseteq B$, $B \not\subseteq C$ and $A \subseteq C$





Possible.

~~Exam~~ $A \not\subseteq B$ means that not true that $(\forall x)(x \in A \Rightarrow x \in B)$ i.e. $(\exists x)(x \in A \wedge x \notin B)$

Example $A = \{2, 3\}$

$B = \{3, 4, 5\}$

$C = \{1, 2, 3, 4\}$

#17 Prove that $X = Y$

Pf: First show that $X \subseteq Y$. Let $x \in X$. Then $x \in \mathbb{N}$ and $x^2 < 30$. This means $x < \sqrt{30} < 6$. Therefore $x \in Y$.

Now show that $Y \subseteq X$. Let $y \in Y$. Then $y = 1, 2, 3, 4$ or 5. In each case $y^2 < 30$. Therefore $y \in X$.

(5) Power Sets.

<u>Def</u>: Let A be a set. Then $P(A)$, the <u>power set</u> of A, is the set of all subsets of A.

<u>eg.</u> $A = \{a, b, c\}$ then

$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \cancel{\{b,a\}}, \{b,c\}, \underline{\{a,b,c\}}\}$

- Note that $\emptyset \in P(A)$ for any set A.
- $P(\emptyset) = \{\emptyset\}$
- Note that $\emptyset \subseteq P(A)$ and $\emptyset \in P(A)$

eg $A = \{a\}$ then $P(A) = \{\emptyset, \underline{\{a\}}\}$

Is $a \in P(A)$? No. $\{a\} \in P(A)$

Is $A \subseteq P(A)$? No. $A \in P(A)$

$A = \{a, b, c\}$ Is $\{\emptyset\} \subseteq A$?

No: If so then $\emptyset \in A$ which is not true.

Thm: If A has n elements, then $\mathcal{P}(A)$ has 2^n elements.

Idea: Can write $A = \{x_1, x_2, \dots, x_n\}$

i.e. I can "index" A by $\{1, 2, \dots, n\}$

Let $C \subseteq A$. How can I determine C?

I can write a list of n 1's and 0's where I put a 1 in kth spot if $x_k \in C$ and a 0 in kth spot if not.

$$\begin{array}{cccccc} \underline{1} & \underline{0} & \underline{0} & \underline{1} & \cdots & \underline{0} \\ \uparrow & \uparrow & \uparrow & \uparrow & & \uparrow \\ 2 & 2 & 2 & 2 & & 2 \\ \text{choices} & & & & & \end{array}$$

So a total of 2^n choices of 1's and 0's.

Russell's Paradox:

Idea is that we can't let the idea of what is a "set" become too large. We have seen that an element of a set can itself be a set (e.g. $\mathcal{P}(A)$). So conceiveable that a set A could satisfy $A \in A$.

(For example suppose A = "the collection of all infinite sets")

In most cases ~~$A \notin A$~~ that we deal with, $A \notin A$. Now consider

$$\underline{X} = \{x : x \notin x\}$$

Is $\underline{X} \in \underline{X}$?

If $\underline{X} \in \underline{X}$ then $\underline{X} \notin \underline{X}$

If $\underline{X} \notin \underline{X}$ then $\underline{X} \in \underline{X}$ $\quad$ <u>paradox</u>

$\underline{X}$ cannot be included in any consistent (i.e. makes sense) ~~set~~ theory of set.

2.2 Set operations

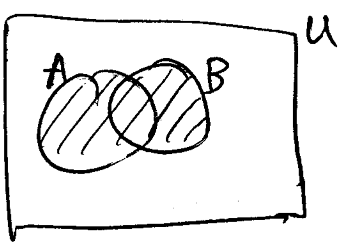
① Union $x \in A \cup B$ if $x \in A$ <u>or</u> $x \in B$

$$\left(A = \{x : P(x)\},\ B = \{x : Q(x)\} \quad A \cup B = \{x : P(x) \vee Q(x)\} \right)$$

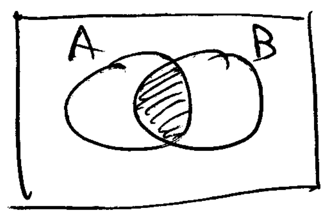
Intersection $x \in A \cap B$ if $x \in A$ <u>and</u> $x \in B$

Difference $x \in A - B$ if $x \in A$ and $x \notin B$

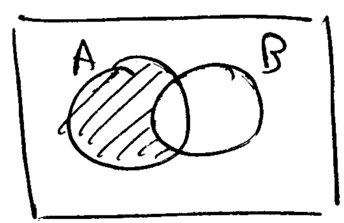
Complement $x \in \tilde{A}$ if $x \notin A$



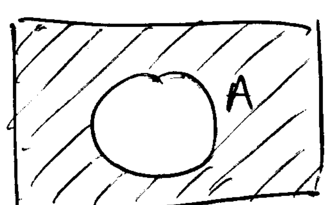
$A \cup B$



$A \cap B$



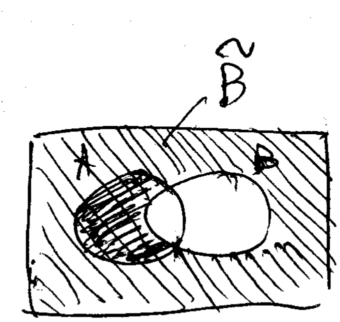
$A - B$



$\tilde{A}$

② Examples.

1. Prove $A - B = A \cap \tilde{B}$

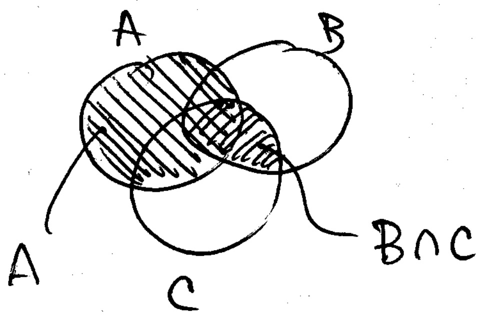


Pf: First show $A-B \subseteq A \cap \tilde{B}$. So let $x \in A-B$. This means that $x \in A$ and $x \notin B$. But the latter means $x \in \tilde{B}$. So $x \in A$ and $x \in \tilde{B}$ so $x \in A \cap \tilde{B}$. Now show $A \cap \tilde{B} \subseteq A-B$. Let $x \in A \cap \tilde{B}$. This means $x \in A$ and $x \in \tilde{B}$. But $x \in \tilde{B}$ means $x \notin B$. Hence $x \in A-B$. Therefore $A-B = A \cap \tilde{B}$.

2. Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

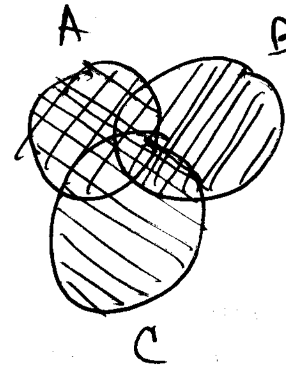
Proof: Let $x \in A \cup (B \cap C)$. If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$. Therefore $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$ then $x \in B$ and $x \in C$. Therefore $x \in A \cup B$ and $x \in A \cup C$. Therefore $x \in (A \cup B) \cap (A \cup C)$. Hence $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$ then $x \in A \cup (B \cap C)$. If $x \notin A$ then since $x \in A \cup B$, $x \in B$ and since $x \in A \cup C$, $x \in C$. Therefore $x \in B \cap C$ so that $x \in A \cup (B \cap C)$. Thus $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.



A

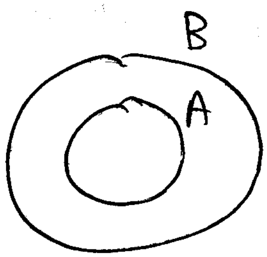
$B \cap C$



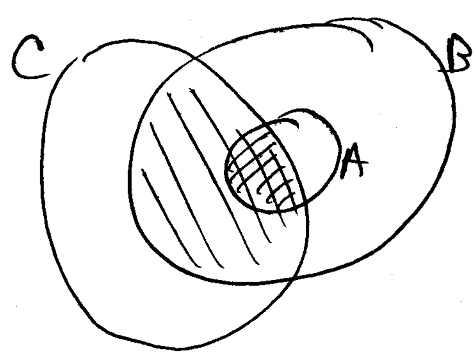
$A \cup B$

$A \cup C$

3. Prove: If $A \subseteq B$ then $A \cap C \subseteq B \cap C$ for all sets C.

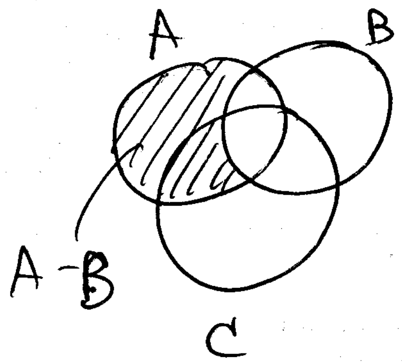


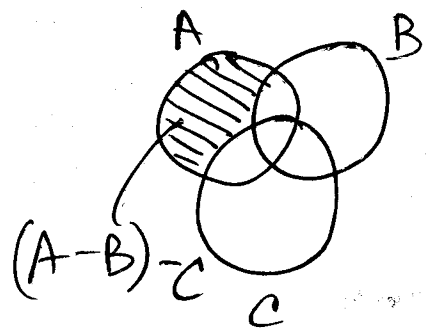
$A \subseteq B$

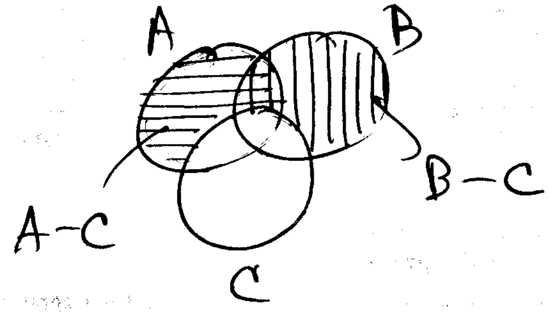


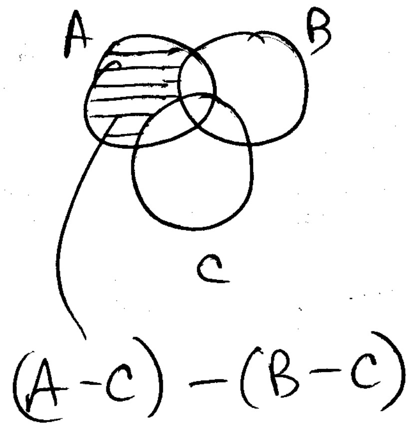
Pf: Assume $A \subseteq B$. Let C be any set and let $x \in A \cap C$. Then $x \in A$ and $x \in C$ so in particular $x \in A$. Since $A \subseteq B$, $x \in B$. Therefore $x \in B$ and $x \in C$ so $x \in B \cap C$.

4. #10 (e) Prove that $(A-B)-C = (A-C)-(B-C)$









$$(A-B)-C = \underline{(A \cap \tilde{B}) \cap \tilde{C}}$$

$$(A-C)-(B-C) = (A \cap \tilde{C}) - (B \cap \tilde{C})$$

$$= (A \cap \tilde{C}) \cap \widetilde{(B \cap \tilde{C})}$$

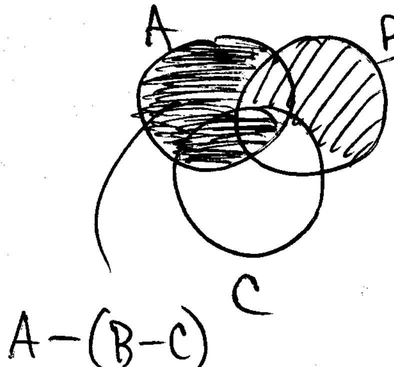
$$= (A \cap \tilde{C}) \cap (\tilde{B} \cup C)$$

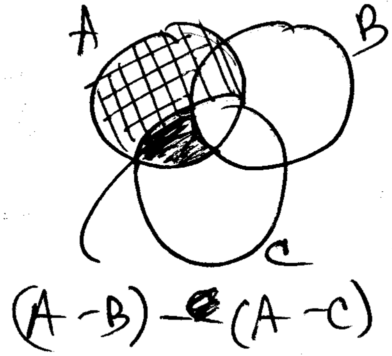
$$= (A \cap \tilde{C} \cap \tilde{B}) \cup \underbrace{(A \cap \tilde{C} \cap C)}_{\emptyset}$$

$$= \underline{A \cap \tilde{C} \cap \tilde{B}}$$

#14 (e) $A-(B-C) = (A-B)-(A-C)$

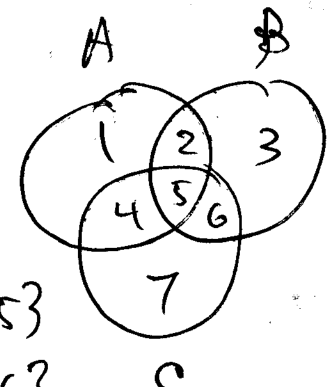
(Find counter example)





/// A-B

||| A-C



$A-(B-C) = \{1,4,5\}$

$(A-B)-(A-C) = \{4\}$.

$A = \{1,2,4,5\}$

$B = \{2,3,5,6\}$

$C = \{4,5,6,7\}$

eg 12 (f) $A \cap B' \subseteq C$