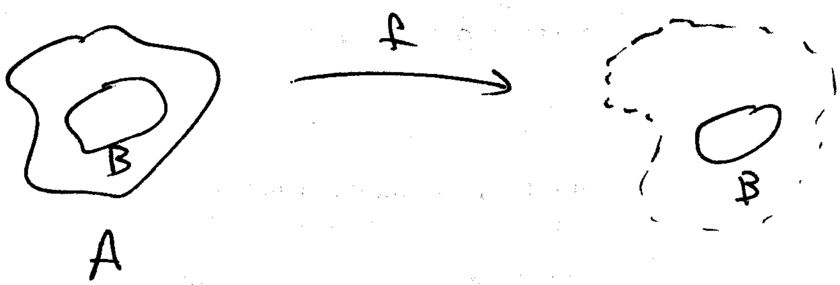
Final Exam – Monday July 20

Cumulative, $\frac{1}{2}$ on 4.4, 5.1–5.3.

Pidgeonhole principle:

If $n, r \in \mathbb{N}$ with $r < n$, and if $f : \mathbb{N}_n \to \mathbb{N}_r$ then f is not one-to-one.

Thm: A finite set is not equivalent to one of its proper subsets.



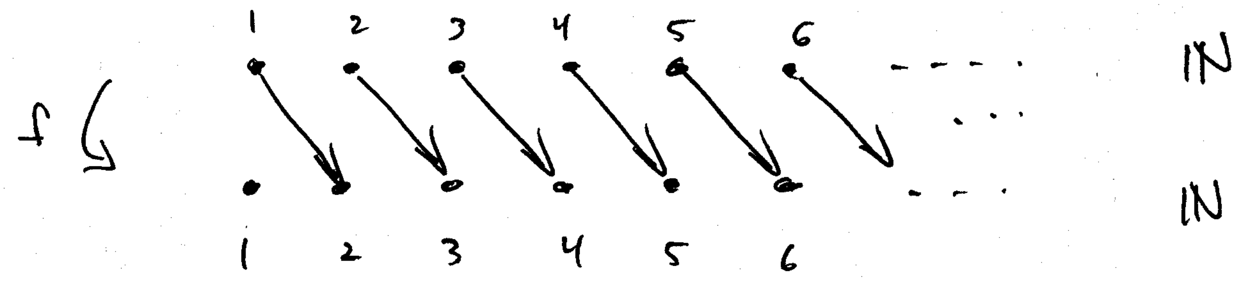
Pf: If A is finite then $A \approx \mathbb{N}_k$ some $k \in \mathbb{N}$. If $B \subsetneq A$ then B is also finite and $B \approx \mathbb{N}_n$ for some $n < k$. If $A \approx B$ then there would be a bijection $f : A \to B$. But if this were true then there would be bijections h and g such that $h : A \to \mathbb{N}_k$ and $g : B \to \mathbb{N}_n$. Hence we would have a bijection $g \circ f \circ h^{-1} : \mathbb{N}_k \to \mathbb{N}_n$. This contradicts Pidgeonhole Principle.

$$\left(\mathbb{N}_k \xrightarrow[h^{-1}]{} A \xrightarrow[f]{} B \xrightarrow[g]{} \mathbb{N}_n\right)$$

Contrapositive: If a set is equivalent to one of its proper subsets then it is infinite.

5.2 Infinite Sets.

Thm: $\mathbb{N}$ is infinite.



Pf: For example, let $f: \mathbb{N} \to \mathbb{N}$ be given by $f(x) = x+1$. Then f is one-to-one and maps $\mathbb{N}$ onto $f(\mathbb{N}) = \{2, 3, 4, \ldots\} \subsetneq \mathbb{N}$.

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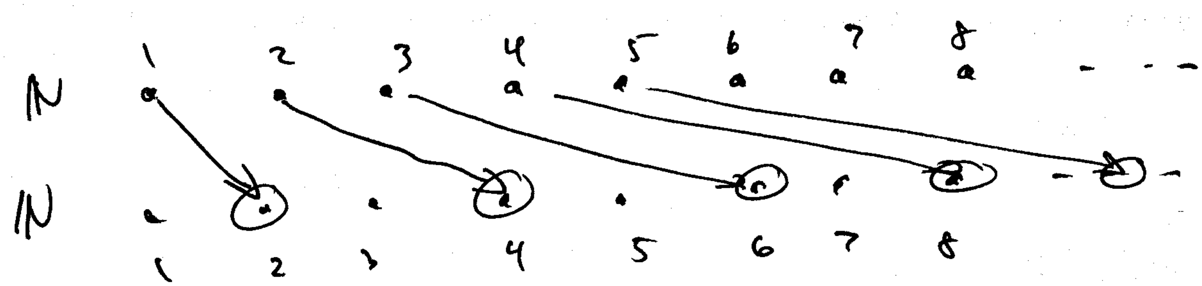
eg. ~~f~~ $f: \mathbb{N} \to \mathbb{N}$ $\quad f(x) = 2x$

f maps $\mathbb{N}$ onto $\{$even numbers in $\mathbb{N}\}$.



Thm: If A is infinite and $A \subseteq B$ then B is infinite.

Pf: Suppose that B is finite and that $A \subseteq B$. Then by previous result A is finite.

Def: A set S is denumerable (or countably infinite) if $S \approx \mathbb{N}$. We say S has cardinality $\aleph_0$

($\aleph_0$ = aleph)

Thm: If $A \cap B = \emptyset$ and A and B are denumerable then $A \cup B$ is denumerable.

Idea:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	...	
•	•	•	•	•	•	•	...	A
1	2	3	4	5	6	7		
•	•	•	•	•	•	•	...	B
b_1	b_2	b_3	b_4	b_5	b_6	b_7		

$A \approx \mathbb{N}$ $f: A \to \mathbb{N}$ bijection $a_1 = \cancel{f(1)}$

also then $g: \mathbb{N} \to A$ bijection $g = f^{-1}$

$a_1 = g(1),\ a_2 = g(2), \dots$

	1	2	3	4	5	6	7	8
$A \cup B$:	•	•	•	•	•	•	•	• ...
	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4 ...

PF: We need to find a bijection $f: \mathbb{N} \to A \cup B$ or a bijection $g: A \cup B \to \mathbb{N}$. Since A & B are denumerable there are bijections $f: A \to \mathbb{N}$ and $h: B \to \mathbb{N}$. Let $e: \mathbb{N} \to \mathbb{N}$ be given by $e(x) = 2x$ and $o: \mathbb{N} \to \mathbb{N}$ by $o(x) = 2x - 1$. Then define $g: A \cup B \to \mathbb{N}$ by

$$g(t) = \cancel{\begin{cases} f \circ e(t) & \text{if } t \in A \\ g \circ o(t) & \text{if } t \in B \end{cases}} = \begin{cases} e \circ f(t) & t \in A \\ o \circ h(t) & t \in B. \end{cases}$$

Coro: $\mathbb{Z}$ is denumerable

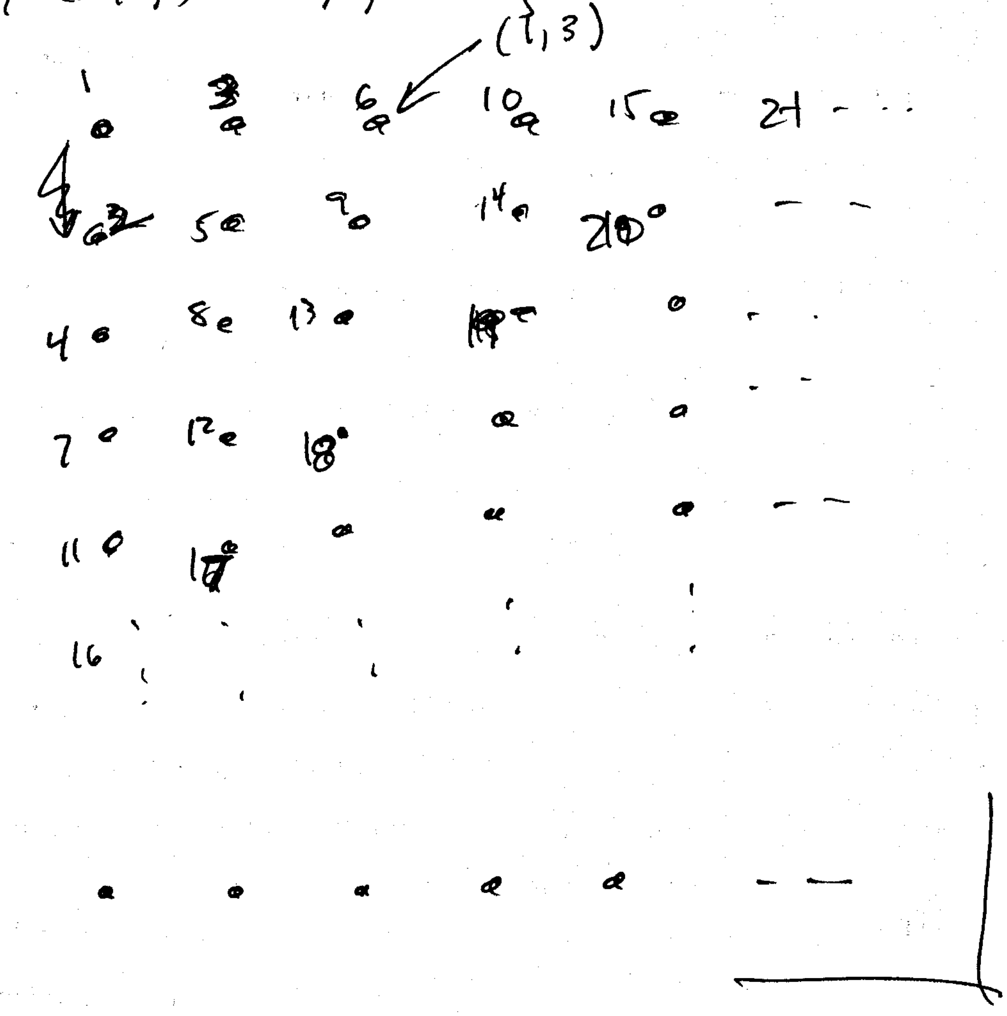
Pf: $\mathbb{Z} = \mathbb{N} \cup \mathbb{N}^- \cup \{0\}$

Thm: $\mathbb{N} \times \mathbb{N}$ is denumerable

$\mathbb{N} \times \mathbb{N} = \{(x,y) : x, y \in \mathbb{N}\}$

Picture:

$\mathbb{N} \times \mathbb{N}$



$\mathbb{N}$

Pf: Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given by $f(x,y) = 2^{x-1}(2y-1)$. Then f is a bijection, so $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$.

Why is f a bijection?

one-to-one: Suppose $f(x_1, y_1) = f(x_2, y_2)$

Then $2^{x_1-1}(2y_1-1) = 2^{x_2-1}(2y_2-1)$

$$2^{x_1-x_2}(2y_1-1) = \underbrace{(2y_2-1)}_{\text{odd}}$$

Since $2y_2-1$ is odd we must have $x_1 = x_2$

This leaves $2y_1-1 = 2y_2-1$ or $y_1 = y_2$.

onto: leave to you.

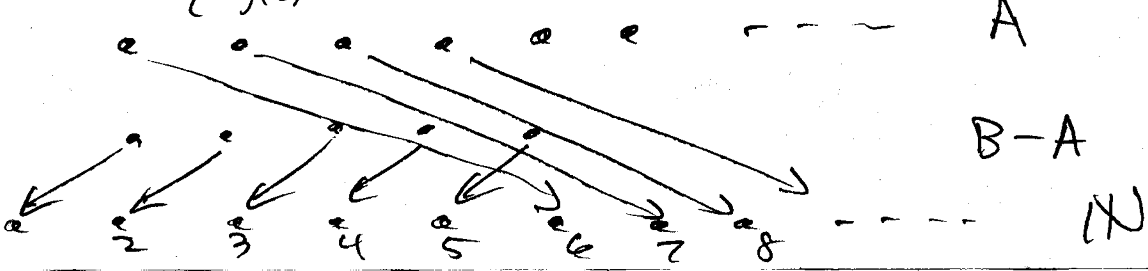
<u>Def</u>: A set S is <u>countable</u> if it is finite or denumerable and <u>uncountable</u> if it is not countable.

<u>Thm</u>: If A is denumerable and B is finite then $A \cup B$ is denumerable.

<u>Pf</u>: Note that $A \cup B = A \cup (B-A)$ and the latter is a disjoint union. Also since $B-A \subseteq B$, it is finite. Since $B-A$ is finite there is a bijection $f: B \to \mathbb{N}_k$ some $k \in \mathbb{N}$ and since A denumerable there is bijection $g: A \to \mathbb{N}$. Define h by

$$h(t) = \begin{cases} f(t) & \text{if } t \in B \\ g(t)+k & \text{if } t \in A. \end{cases}$$

<u>Idea</u>:



Thm: If A and B are countable then $A \cup B$ is countable.

Thm: The interval $(0,1)$ is uncountable.

Pf: (1) Each $a \in (0,1)$ has a decimal expansion

$$a = .a_1 a_2 a_3 a_4 \cdots \qquad a_k \in \{0,1,\ldots,9\}$$

This expansion is not unique since e.g.

$$.4859999\cdots = .486000\cdots$$

so we say a decimal expansion is in normalized form if it does not end in repeated 9s.

So every $a \in (0,1)$ has a unique normalized decimal expansion.

(2) Since, for example, $\{\frac{1}{n} : n \in \mathbb{N}\} \subseteq (0,1)$

$(0,1)$ is not finite.

(3) Suppose that $(0,1)$ were denumerable. This would mean there is a bijection $f: \mathbb{N} \to (0,1)$

Will show no such f exists.

if it did, we could list the elements of $(0,1)$

$$\begin{array}{ll} f(1) & .a_{11}\,a_{12}\,a_{13}\,a_{14}\cdots \\ f(2) & .a_{21}\,a_{22}\,a_{23}\,a_{24}\cdots \\ f(3) & .a_{31}\,a_{32}\,a_{33}\,a_{34}\cdots \\ f(4) & .a_{41}\,a_{42}\,a_{43}\,a_{44}\cdots \\ \vdots & \end{array}$$

Cantor's diagonalization argument.

Let $b_i = \begin{cases} 5 & \text{if } a_{ii} \neq 5 \\ 3 & \text{if } a_{ii} = 5 \end{cases}$ and let

$b = .b_1 b_2 b_3 b_4 \cdots$

Then b is not on the list because b differs from $f(n)$ in the n^{th} decimal place (at least). Hence f is not onto.

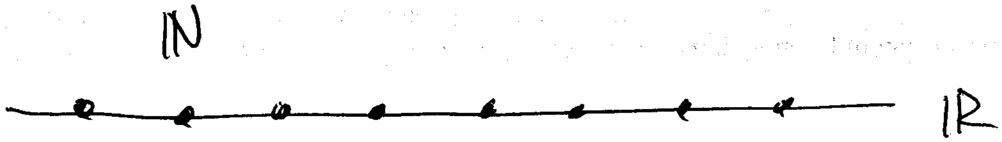
<u>Def</u>: A set S has <u>cardinality</u> c if $S \approx (0,1)$

5.3 Countable Sets.

Thm: The set of rational numbers $\mathbb{Q}$ is denumerable.

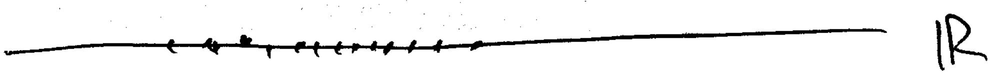
Idea: (1) We know that $\mathbb{N} \subseteq \mathbb{R}$

countable ↗ ↑ uncountable

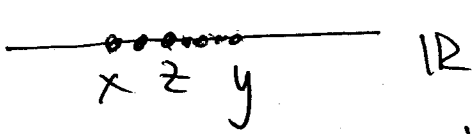


Intuitively $\mathbb{N}$ is "sparse" in $\mathbb{R}$, i.e. it has gaps.

(2) Not true of $\mathbb{Q}$. $\mathbb{Q}$ is <u>dense</u> in $\mathbb{R}$



Thm: Let $x, y \in \mathbb{Q}$ with $x < y$. Then there is a $z \in \mathbb{Q}$ with $x < z < y$



Pf: If $x = \frac{m}{n}$ and $y = \frac{p}{q}$ then let $z = \frac{x+y}{2} = \frac{mq+np}{2nq} \in \mathbb{Q}$

$\mathbb{Q}$ has <u>no</u> gaps. But it has the same number of elements as $\mathbb{N}$.

Idea of Proof: Think first of $\mathbb{Q}^+ = \{x \in \mathbb{Q} : x \geq 0\}$

each $x \in \mathbb{Q}$ can be written $x = \frac{p}{q}$ $p, q \in \mathbb{N}$.

List $\mathbb{Q}$ as follows:

$$
\begin{array}{ccccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \frac{6}{1} & \cdots \\[6pt]
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \cdots \\[6pt]
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} & \cdots \\[6pt]
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} & \cdots \\[6pt]
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \frac{6}{5} & \cdots
\end{array}
$$