Exam 6 ~~3.4~~ 3.5, 4.1–4.3 Wednesday

Final Exam 7/20 Monday ← 430–715 – cumulative

emphasizes (about half) sections 4.4 + later.

4.1 8 (b) (c)

~~(b)~~ ~~(c)~~ (b) $f: \mathbb{Z}_6 \to \mathbb{Z}_6$ $f(\overline{x}) = [x+1]$

$\mathbb{Z}_6 = \mathbb{Z}/\equiv_6$ $\overline{0} = \{0, 6, -6, 12, -12, 18, -18, \ldots\}$

$x \equiv_6 y$ iff $6 | (x-y)$ $\overline{1} = \{1, 7, -5, 13, -11, \ldots\}$

$\vdots$

Is f a function?

If $f(\overline{x}) = [y]$ and $f(\overline{x}) = [z]$ then $[y] = [z]$

~~$\overline{2} = \{x \equiv 6m2 \; m \in \mathbb{Z}\}$~~ Problem: Can have $\overline{x_1} = \overline{x_2}$ but $x_1 \neq x_2$

$f(\overline{2}) = [3]$

Verify: if $\overline{x_1} = \overline{x_2}$ then $[x_1+1] = [x_2+1]$

$\overline{x_1} = \overline{x_2}$ means $x_1 = x_2 + 6k$ some $k \in \mathbb{Z}$.

$[x_1+1] = [x_2+1]$ means $x_1 + 1 = x_2 + 1 + 6n$ some $n \in \mathbb{Z}$.

Must show: If $x_1, x_2 \in \mathbb{Z}$ and $x_1 = x_2 + 6k$ some $k \in \mathbb{Z}$ then $x_1 + 1 = x_2 + 1 + 6n$ some $n \in \mathbb{Z}$.

To prove just take $n = k$.

So f is a function.

8 (d) $f: \mathbb{Z}_4 \to \mathbb{Z}_6$ $\quad f(\overline{x}) = [2x+1]$

$\overline{x_1} = \overline{x_2}$ iff $x_1 = x_2 + 4k$ some $k \in \mathbb{Z}$

$[2x_1+1] = [2x_2+1]$ iff $2x_1 + 1 = 2x_2 + 1 + 6n$ some $n \in \mathbb{Z}$

$\cancel{2x_1+1} \quad 2(x_2+4k)+1 = 2x_2+1+6n$

$\cancel{2x_2} + 8k + \cancel{1} = \cancel{2x_2} + \cancel{1} + 6n$

Need $\underline{8k = 6n}$. That is, Given $k \in \mathbb{Z}$, can I always find $n \in \mathbb{Z}$ such that $6n = 8k$?

Say $k=2$: Solve $6n = 16$. No such $n \in \mathbb{Z}$.

This means: Say $x_1 = 0 \quad x_2 = 8$

So $\overline{0} = \overline{-8}$

but $[1] \overset{?}{=} [-15]$ $\underline{\text{NO}}$.

$[1] = \{1, 7, -5, 13, -11, 19, -17, \dots\}$

$[-15] = \{-15, -9, -21, -3, -27, \dots\}$

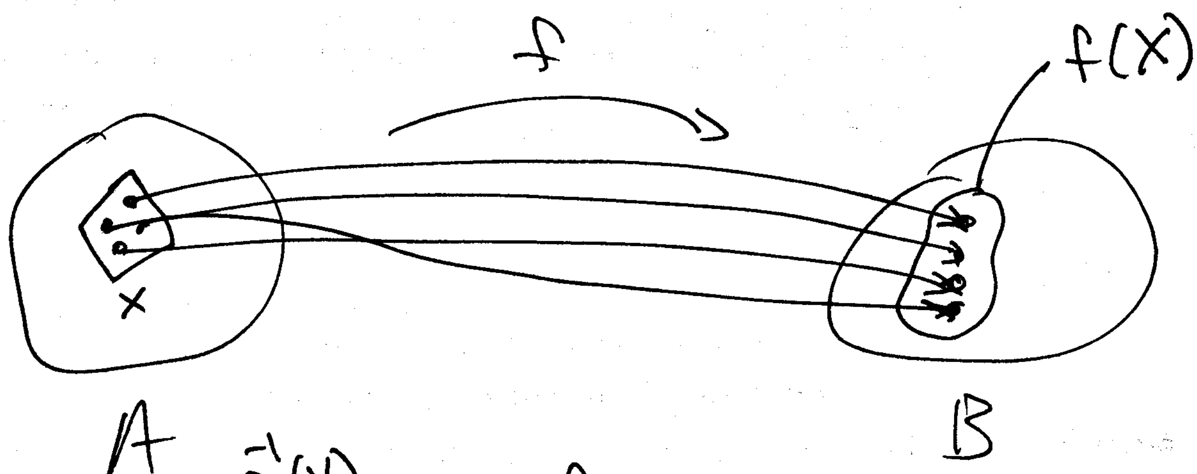
## 4.4 Images of Sets.

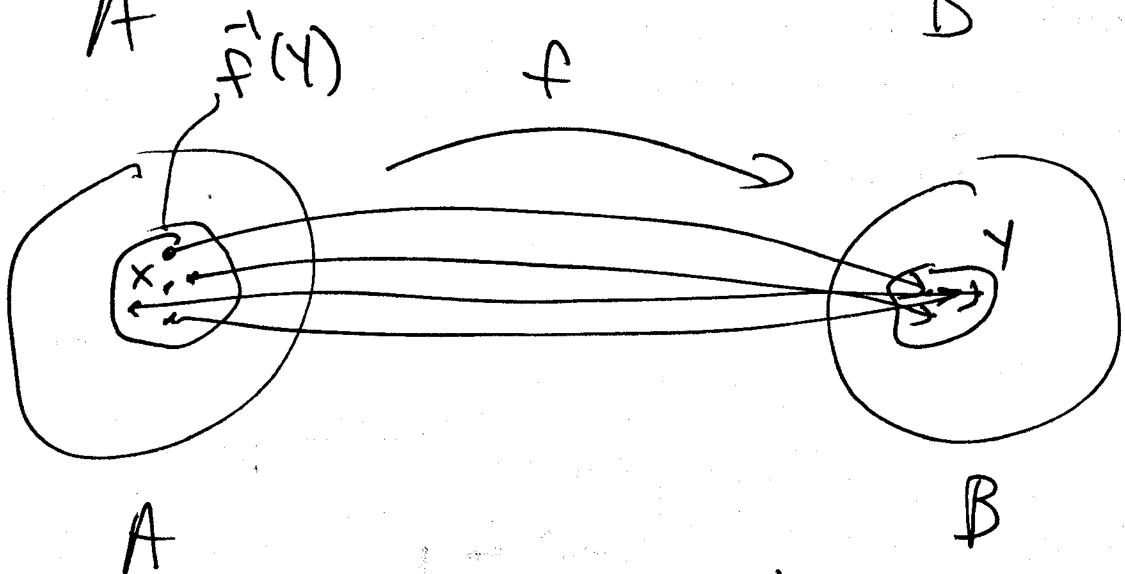
**Def:** Let $f: A \to B$, let $X \subseteq A$, $Y \subseteq B$.
The image of $X$ under $f$ is the set

$$f(X) = \{y \in B : f(x) = y \text{ some } x \in X\}$$

The inverse image of $Y$ under $f$ (or pre-image of $Y$) is the set

$$f^{-1}(Y) = \{x \in A : f(x) = y \text{ for some } y \in Y\}$$
$$= \{x \in A : f(x) \in Y\}.$$




— Note: Talking about $f^{-1}(Y)$ does not mean $f^{-1}$ exists as a function.

– $f$ maps $X$ onto $f(X)$

Does $f$ map $f^{-1}(Y)$ onto $Y$? Not necessarily

Thm: Let $f: A \to B$, $D \subseteq A$, $E \subseteq B$

Then for all $a \in A$,

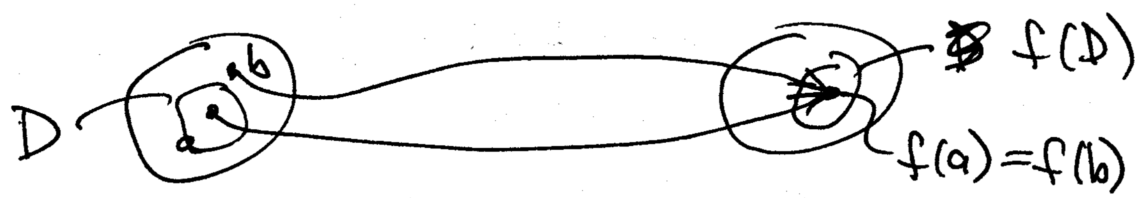
(i) $a \in f^{-1}(E)$ if and only if $f(a) \in E$

(ii) if $a \in D$ then $f(a) \in f(D)$ (converse does not hold)

Pf: (i) Suppose $a \in f^{-1}(E)$. This means there is a $y \in E$ such that $f(a) = y$ or $f(a) \in E$. Suppose that $f(a) \in E$. But this is the definition of $a \in f^{-1}(E)$.

(ii) Suppose $a \in D$. If $y = f(a)$ then $y \in f(D)$ since $a \in D$. Hence $f(a) \in f(D)$.

Suppose $f(a) \in f(D)$. Must $a \in D$? $f(a) \in f(D)$ only means there is a $b \in D$ such that $f(b) = f(a)$. Must $b = a$? NO. Unless $f$ is one-to-one



Thm: Let $f: A \to B$, $X \subseteq A$, $Y \subseteq B$

(i) $X \subseteq f^{-1}(f(X))$

(ii) $f(f^{-1}(Y)) \subseteq Y$

Pf: (i) Let $x \in X$ then $f(x) \in f(X)$. But this means $x \in f^{-1}(f(X))$.

(ii) Let $y \in f(f^{-1}(Y))$. This means there is an $x \in f^{-1}(Y)$ such that $f(x) = y$. But $x \in f^{-1}(Y)$ means $f(x) \in Y$, that is, $y \in Y$.

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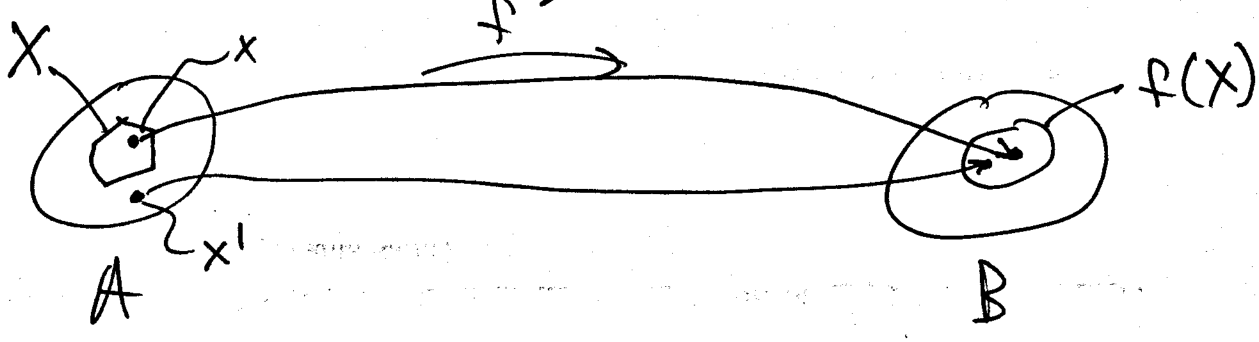
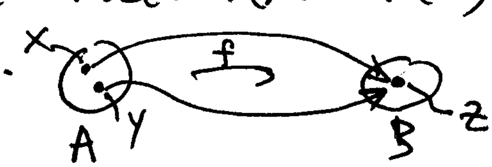
What about equality?

(i) How could $X \subsetneq f^{-1}(f(X))$?



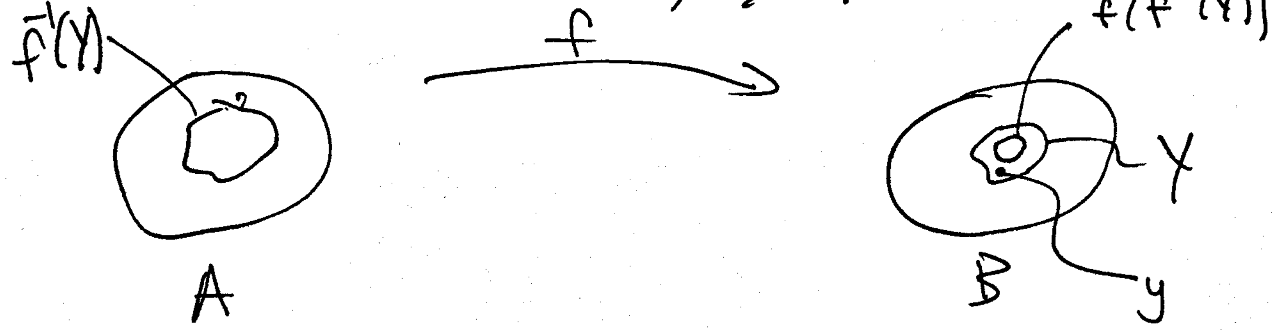

$f^{-1}(f(X))$ is everything in A that "lands" in $f(X)$.
If there is some $x' \in A$, $x' \notin X$ such that $f(x') \in f(X)$ then we have a proper subset.



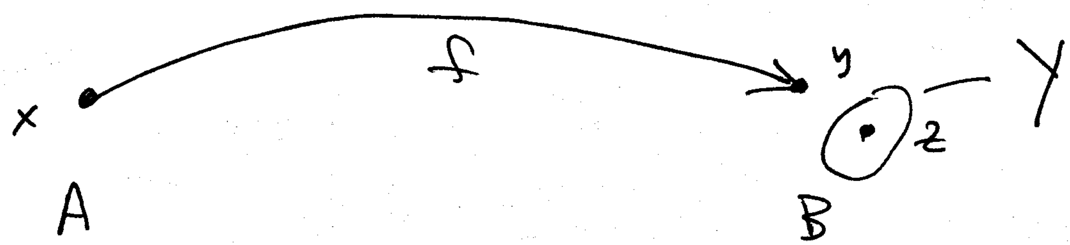

Example: $A = \{x, y\}$ $B = \{z\}$

$f = \{(x,z), (y,z)\}$ If $X = \{x\}$ then $f(X) = \{z\}$ but $f^{-1}(f(X)) = \{x, y\}$

(ii) How could $f(f^{-1}(Y)) \subsetneq Y$ ?




There would have to be some $y \in Y$ not hit by anything in $f^{-1}(Y)$. This can happen only if there is a $y \in Y$ that is not hit by anything in $A$. That is, $f$ is not <u>onto</u>.




$A = \{x\} \quad B = \{y, z\}$

$f = \{(x, y)\}$ Let $Y = \{z\}$. Then $f^{-1}(Y) = \emptyset$

so $f(f^{-1}(Y)) = f(\emptyset) = \emptyset$ and

$f(f^{-1}(Y)) \subsetneq Y$ since $\emptyset \subsetneq \{z\}$.

<u>Thm:</u> Let $f: A \to B$, $X \subseteq A$, $Y \subseteq B$.

(i) $X \subseteq f^{-1}(f(X))$ and $X = f^{-1}(f(X))$ if $f$ is one-to-one

(ii) $f(f^{-1}(Y)) \subseteq Y$ and $f(f^{-1}(Y)) = Y$ if $f$ is onto.

# 5.1 Equivalent Sets.

Idea: Learn how to understand, deal with infinite sets.

① We say $f: A \to B$ is a bijection if it is one-to-one and onto.

This means that $f^{-1}: B \to A$ is also a bijection.

Also say $A$ and $B$ are in one-to-one correspondence.

In this case, $A$ and $B$ are "identical" as sets.

Def: Two sets $A$ and $B$ are equivalent if there exists a bijection $f: A \to B$.

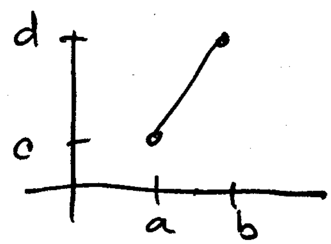
We write $A \approx B$

e.g. $A = \{a, b, c\}$ $B = \{1, 2, 3\}$

$f = \{(a,3), (b,1), (c,2)\}$

e.g. given $a < b$ and $c < d$. Then $(a,b) \approx (c,d)$

$(|b-a| \neq |d-c|)$



Line from $(a,c)$ to $(b,d)$

$m = \dfrac{d-c}{b-a}$ $\quad y - c = \dfrac{d-c}{b-a}(x-a)$

$f(x) = \dfrac{d-c}{b-a}(x-a) + c$

- $\approx$ is an equivalence relation

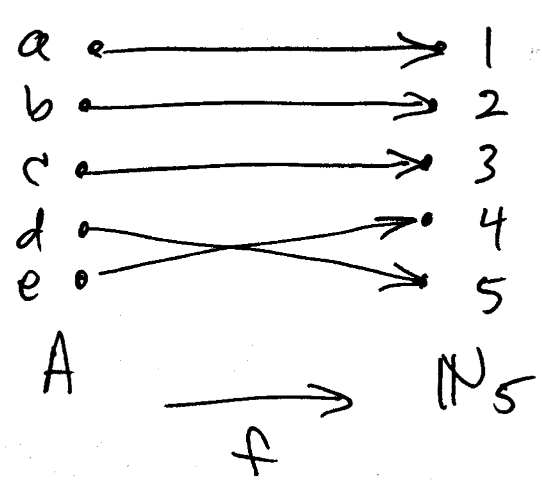
- Given $A$ with $\overline{\overline{A}} = k$ some $k \in \mathbb{N}$
what is $A/\approx$ ? ~~A~~ Every set with $k$ elements.

- we choose a "standard element" of $A/\approx$ , $\mathbb{N}_k = \{1, 2, \dots, k\}$.

<u>Def</u>: We say a set $S$ is <u>finite</u> if $S = \emptyset$ or there exists $k \in \mathbb{N}$ such that $S \approx \mathbb{N}_k$.
$S$ is <u>infinite</u> if it is not finite.
If $S \approx \mathbb{N}_k$ we say $S$ has <u>cardinality</u> $k$
$S = \emptyset$ has cardinality $0$.

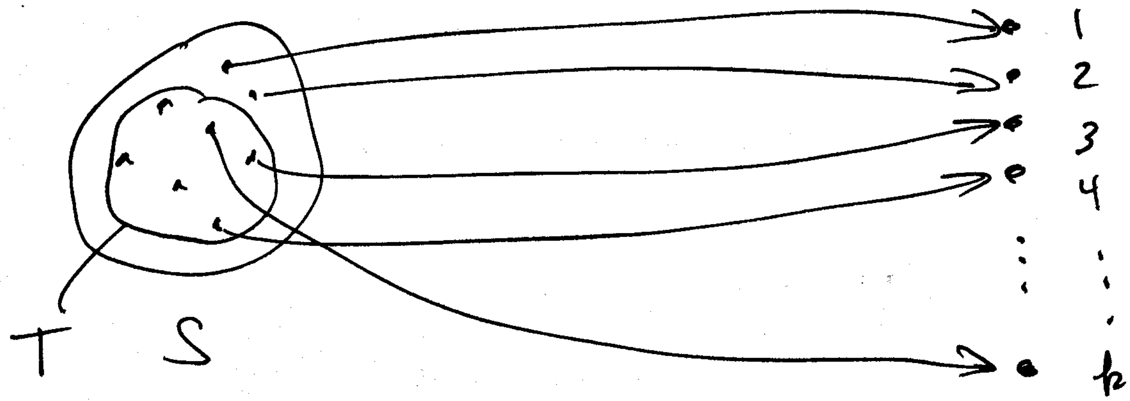
- there are many infinities

- $A = \{a, b, c, d, e\}$ is finite because



Want to show: If $S$ is finite and $T \subseteq S$ then $T$ is finite.

<u>Idea</u>: Suppose $S$ is finite. Then exists $k \in \mathbb{N}$ s.t. $S \approx \mathbb{N}_k$, i.e. exists $f : S \to \mathbb{N}_k$ a bijective function. Need to show exists a bijection $g : T \to \mathbb{N}_n$ some $n \in \mathbb{N}$.




$f(S) = \mathbb{N}_k$

$\to f(T) \subseteq \mathbb{N}_k$

Easiest: Choose for $g$ the function $f|_T$

~~Need to know that~~ then $g : T \to f(T)$ is a bijection so $T \approx f(T) \subseteq \mathbb{N}_k$.

Need to know that any subset of $\mathbb{N}_k$ is finite.

<u>Lemma</u>: For all $k \in \mathbb{N}$, every subset of $\mathbb{N}_k$ is finite.

<u>Pf</u>: Try induction.

(i) $k=1$: If $A \subseteq \mathbb{N}_1 = \{1\}$ then $A = \emptyset$ or $A = \{1\}$. Finite in both cases.

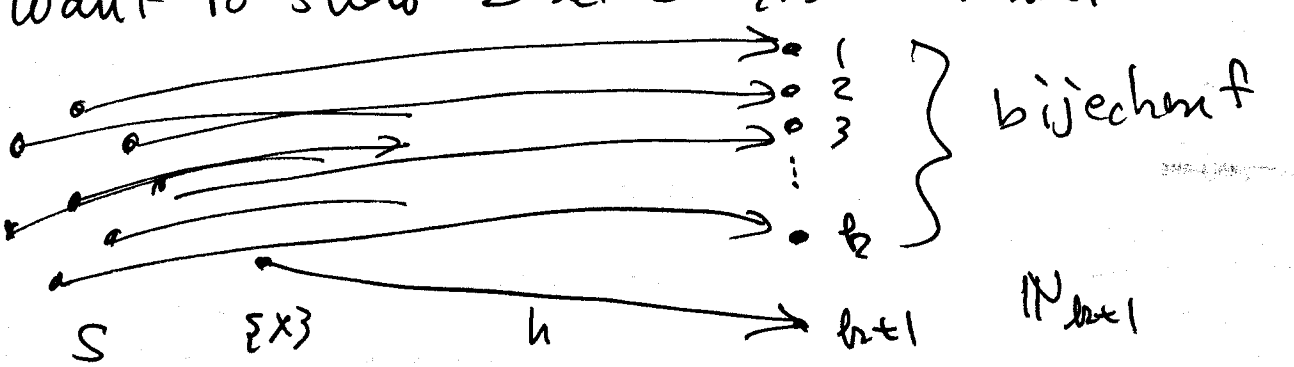
(ii) Suppose result holds for $\mathbb{N}_k$, and let $A \subseteq \mathbb{N}_{k+1}$. Two cases: If $k+1 \notin A$ then $A \subseteq \mathbb{N}_k$ and is finite by induction hypothesis. If $k+1 \in A$ then let $B = A - \{k+1\}$. Then $B \subseteq \mathbb{N}_k$ and is finite by induction hypothesis and $A = B \cup \{k+1\}$.

Need to know that adding one element to a finite set gives a finite set.

<u>Lemma</u>: Let $S$ be finite and let $x \notin S$. Then $S \cup \{x\}$ is finite.

<u>Pf</u>: If $S = \emptyset$ then $S \cup \{x\} = \{x\}$ and $\{x\}$ is finite. If $S \neq \emptyset$ then $S \approx \mathbb{N}_k$ some $k \in \mathbb{N}$.

Want to show $S \cup \{x\} \approx \mathbb{N}_{k+1}$



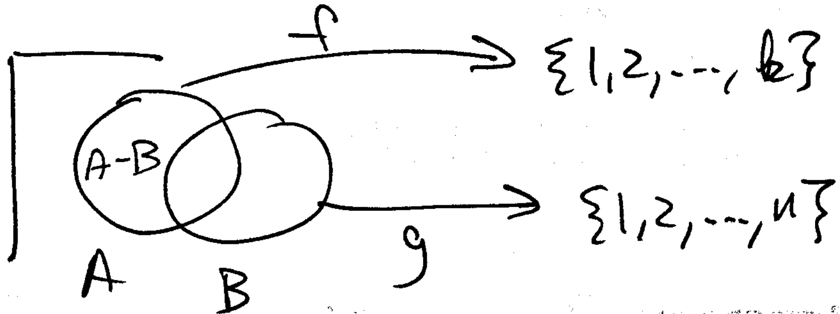
Let $f: S \to \mathbb{N}_k$ be a bijection and define $h: \{x\} \to \{k+1\}$ by $h(x) = k+1$. Then let

$$g(t) = \begin{cases} f(t) & \text{if } t \in S \\ h(t) & \text{if } t = x \end{cases},$$ and $g$ is a bijection

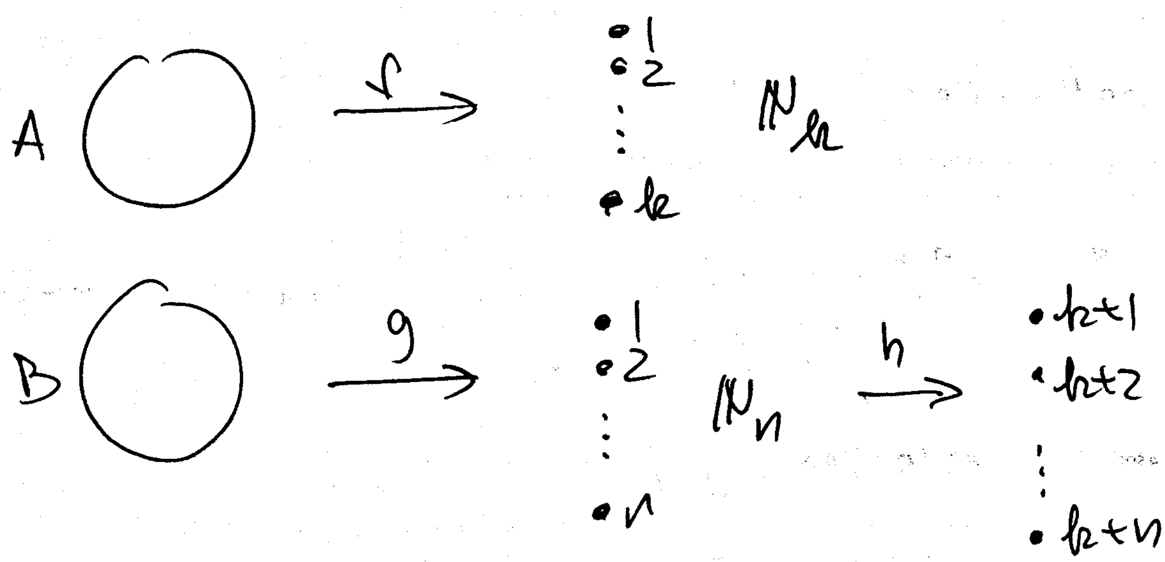
from $S \cup \{x\} \to \mathbb{N}_{k+1}$.

<u>Thm</u>: If $A$ and $B$ are finite then $A \cup B$ is finite.

<u>Pf</u>:




Easier if $A \cap B = \emptyset$



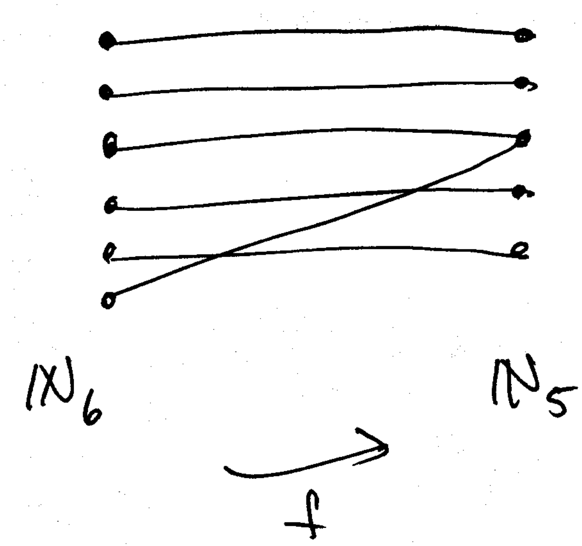

$f, g, h$ bijections

Define
$$r(t) = \begin{cases} f(t) & \text{if } t \in A \\ h \circ g(t) & \text{if } t \in B \end{cases}$$

Then $r: A \cup B \to \mathbb{N}_{k+n}$ is a bijection.

If $A, B$ not disjoint observe that
$A \cup B = (A-B) \cup B$ and these are disjoint.
Since $A - B \subseteq A$ and $A$ is finite, so is $A-B$.
So by previous argument $(A-B) \cup B = A \cup B$
is finite.

<u>Thm</u> (Pidgeonhole principle)
Let $n, r \in \mathbb{N}$ with $r < n$. If $f: \mathbb{N}_n \to \mathbb{N}_r$
then $f$ is not one-to-one




eg #20 (b) $\mathbb{N}_{99} = \{1, 2, 3, \ldots\ldots, 97, 98, 99\}$
Largest possible sum is $99+98+97+\cdots+90 < 1000$
Pick $S \subseteq \mathbb{N}_{99}$ with 10 elements.
There are $2^{10} = 1024$ subsets of $S$. Actually
1023 non-empty subsets of $S$.
So at least 2 subsets of $S$ must add to
same number. Eliminate common elements
to complete proof.