Functions (cont'd)

A function is a relation between 2 sets A and B.

Every $x \in A$ is related to one and only one element of B.

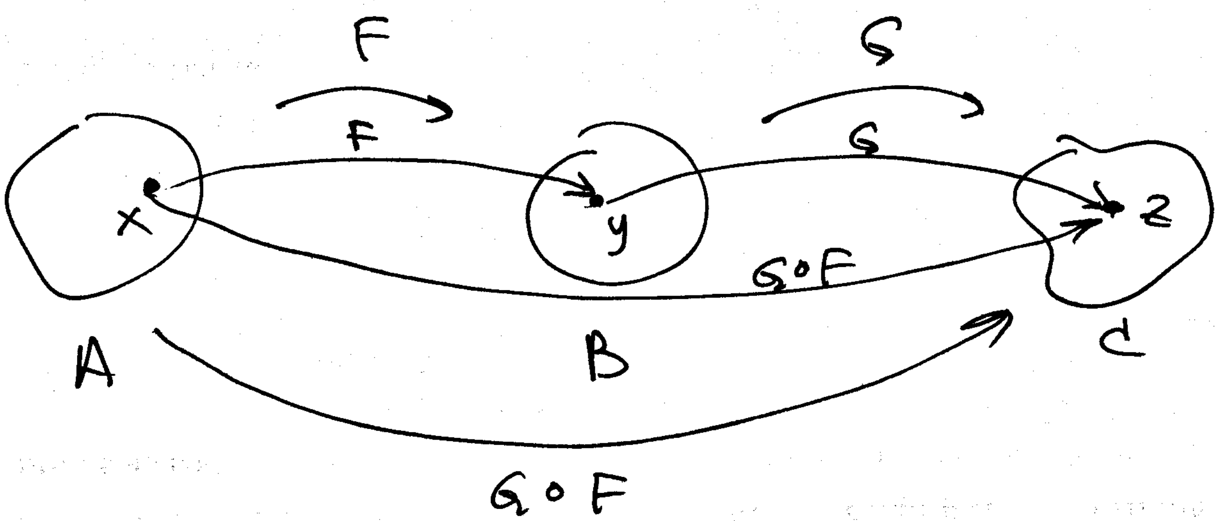
<u>OR</u> If $(x,y) \in f$ and $(x,z) \in f$ then $y = z$

② Composites

If $F: A \to B$ and $G: B \to C$ are functions

define $G \circ F$ by

$$G \circ F = \{(x,z) : \exists y \in B \text{ such that } (x,y) \in F \text{ and } (y,z) \in G\}$$



— If F and G are defined by rules, then $G \circ F$ is written as a rule in the usual way.

<u>eg.</u> $F: \mathbb{R} \to \mathbb{R}$ $\quad F(x) = 2x+1$

$y = 2x+1$

$x \mapsto 2x+1$

"maps to"

$G: \mathbb{R} \to \mathbb{R}$ $\quad G(x) = x^2$

$y = x^2$

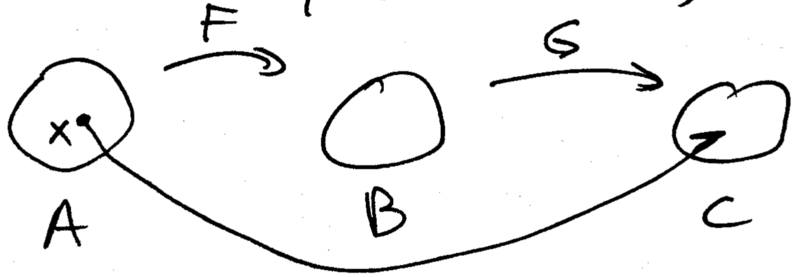
$x \mapsto x^2$

Then $G \circ F(x) = G(F(x)) = (2x+1)^2$

$F \circ G(x) = F(G(x)) = 2(x^2)+1 = 2x^2+1$

Q. Is $G \circ F$ always a function? YES

(i.e. assuming G and F are)

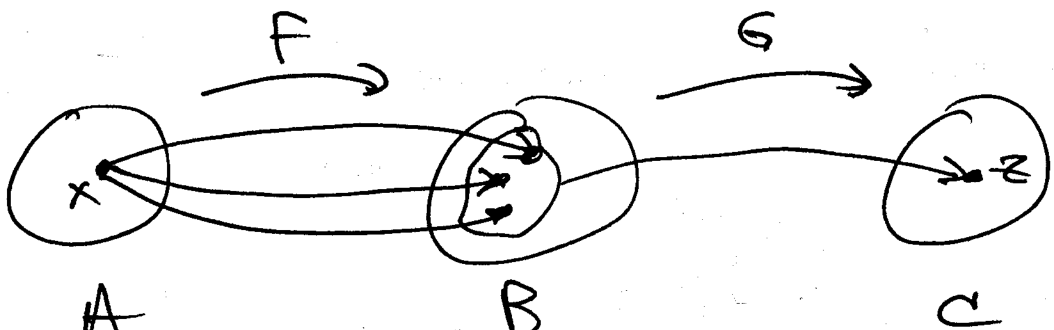


<u>Thm</u>: If $F: A \to B$ and $G: B \to C$ then $G \circ F: A \to C$ is a function.

<u>Pf</u>: Suppose that $(x,y) \in G \circ F$ and $(x,z) \in G \circ F$. Want to show $y = z$. Since $(x,y) \in G \circ F$ there is a $u \in B$ such that $(x,u) \in F$ and $(u,y) \in G$ and since $(x,z) \in G \circ F$ there is a $v \in B$ such that $(x,v) \in F$ and ~~(x,y)~~ $(v,z) \in G$. Since F is a function and (x,u) and $(x,v) \in F$ then $u = v$. Hence $(u,y) \in G$ and $(v,z) \in G$ and $u = v$ implies $y = z$ since G is a function.

Q. Is it possible for $G \circ F$ to be a function while G or F (or both) fail to be functions?

<u>Idea</u>



Suppose G is a constant function, i.e. $G(y) = z$ for all $y \in B$ and some fixed $z \in C$. Then $G \circ F$ can be a function while F is not.

e.g. $A = \{1,2,3\}$ $B = \{a,b,c\}$ $C = \{\alpha\}$

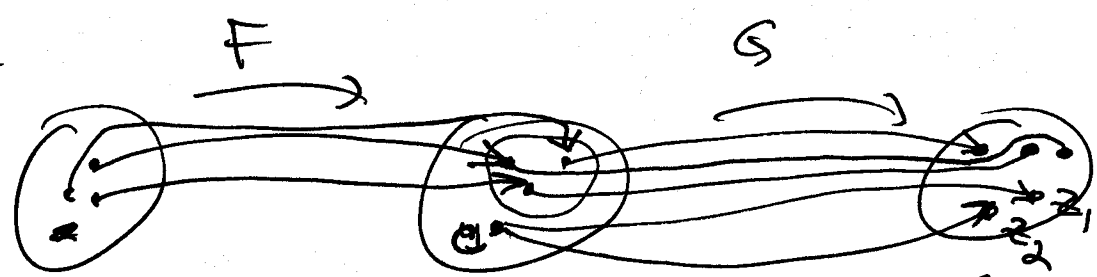
$F = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),(3,a)\}$ not a function

$G = \{(a,\alpha),(b,\alpha),(c,\alpha)\}$ ← function

$G \circ F = \{(1,\alpha),(2,\alpha),(3,\alpha)\}$ ← function

If G is not a function:

idea



$A = \{1,2,3\}$ $B = \{a,b,c\}$ $C = \{\alpha,\beta,\gamma\}$

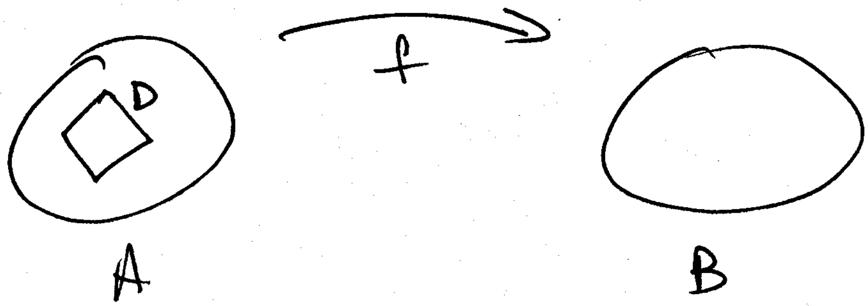
$F = \{(1,a),(2,b),(3,a)\}$

$G = \{(a,\alpha)\ (b,\beta),(c,\alpha),(c,\beta),(c,\gamma)\}$ not a function

$G \circ F = \{(1,\alpha),(2,\beta),(3,\alpha)\}$ ← function

can have both F and G not functions, but $G \circ F$ a function.

(3) Restrictions and Extensions



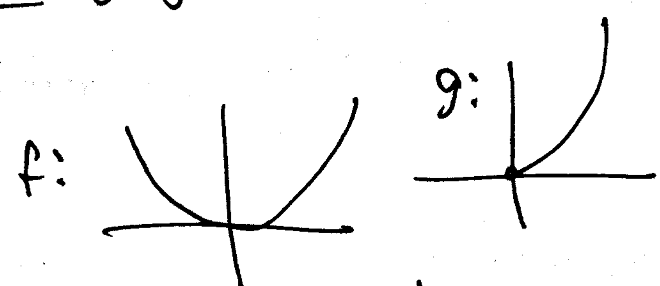
$f: A \to B$ and $D \subseteq A$ then

$$f|_D = \{(x,y) \in f : x \in D\}$$

is the restriction of f to D.

Suppose that $g: D \to B$. If $h: A \to B$ is such that $h(x) = g(x)$ whenever $x \in D$ then h is an extension of g.

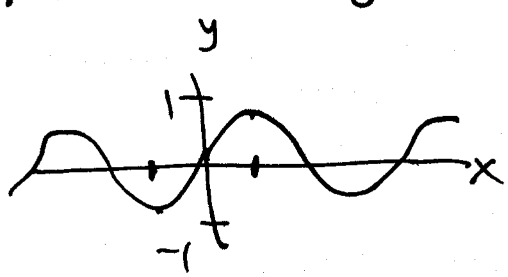
e.g. $f(x) = x^2$, $x \in \mathbb{R}$

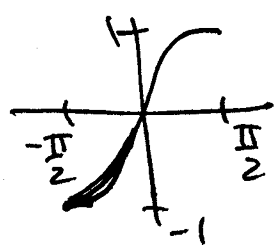


$g(x) = x^2$, $x \geq 0$ is the restriction of f to $[0,\infty)$. Write $g = f|_{[0,\infty)}$

$h(x) = \sin(x)$, $x \in \mathbb{R}$

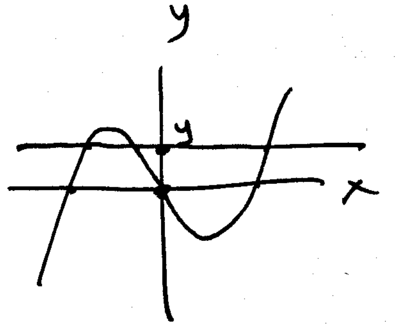
$r(x) = \sin(x)$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$





4.3 One-to-one and onto functions

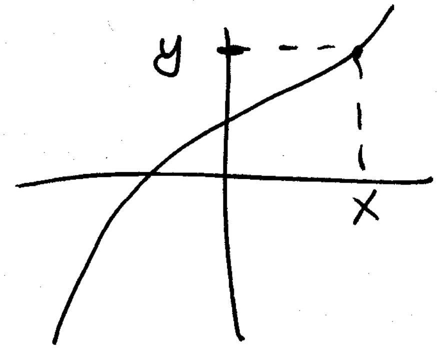
① One-to-one



$y = f(x)$

not one-to-one

for some y there are several solutions to $y = f(x)$



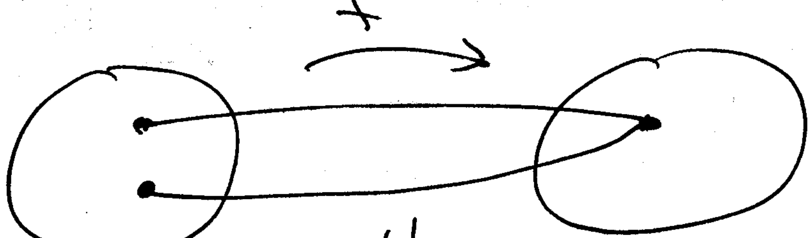
$y = f(x)$

is one-to-one

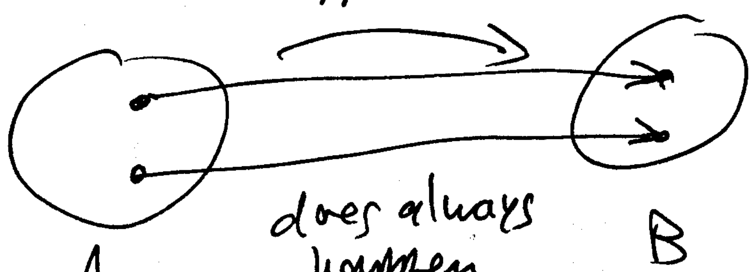
one-to-one means each y corresponds to at most one x in the formula $y = f(x)$.

More precisely: A function $f: A \to B$ is one-to-one iff for all $x, y \in A$, $f(x) = f(y)$ implies $x = y$.

Picture



A can't happen B



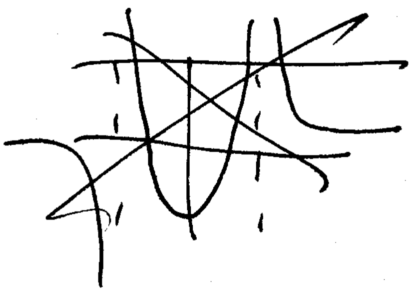
A does always happen B

– Contrapositive: $\forall x, y \in A$, $x \neq y$ implies $f(x) \neq f(y)$.

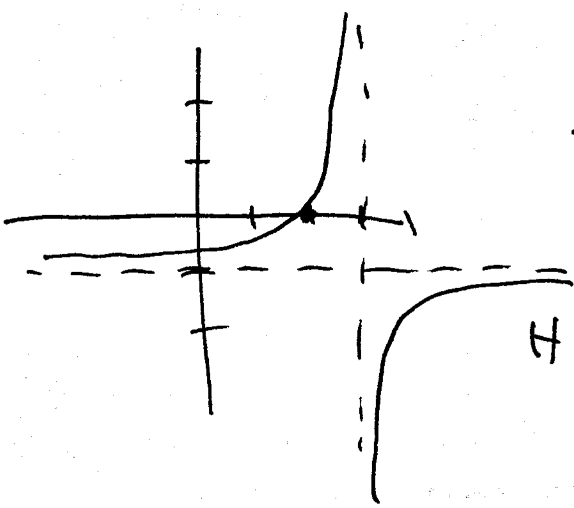
Can say "one-to-one" really means "two-to-two"

\#1 (k) $f: [2,3) \to [0,\infty)$ $\quad f(x) = \dfrac{x-2}{3-x}$

one-to-one? Yes.



Pf: Suppose that $x, y \in [2,3)$ and that $\dfrac{x-2}{3-x} = \dfrac{y-2}{3-y}$. Then we calculate:



$$(x-2)(3-y) = (y-2)(3-x)$$

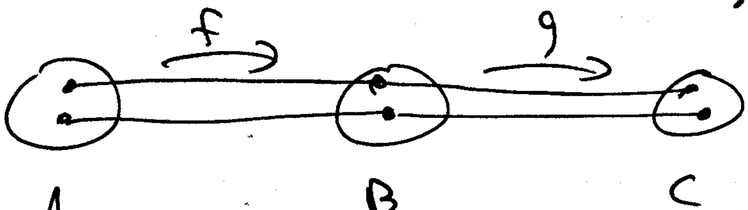
$$3x - \cancel{6} - \cancel{xy} + 2y = 3y - \cancel{6} - \cancel{xy} + 2x$$

$$3x + 2y = 3y + 2x$$

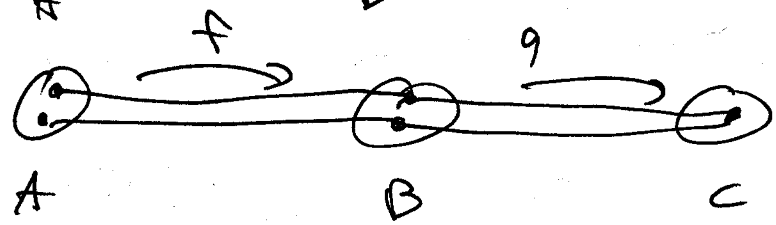
$$x = y$$

Hence $f(x)$ is one-to-one.

\#8 (d) $f: A \to B$ one-to-one $\quad g \circ f$ is not one-to-one



← If g and f are one-to-one, so is $g \circ f$.

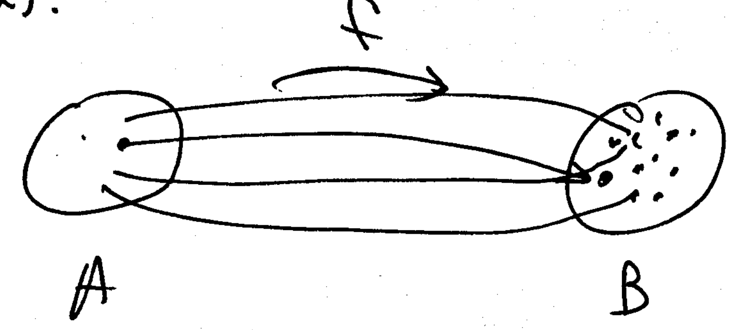


$A = \{1, 2\} \quad B = \{a, b\} \quad C = \{\alpha\}$

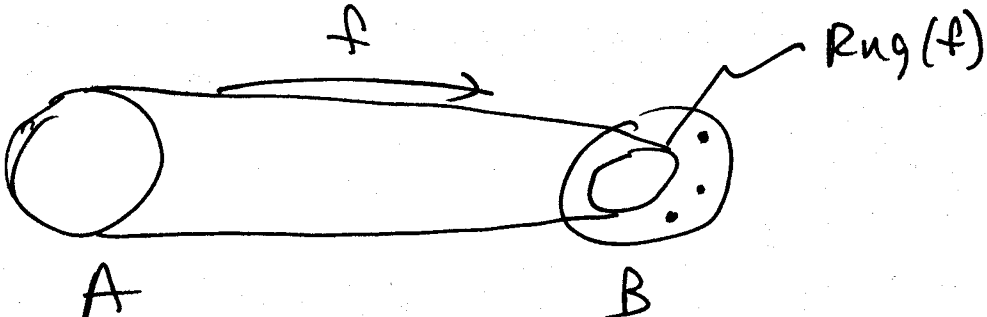
$f = \{(1,a), (2,b)\} \quad g = \{(a,\alpha), (b,\alpha)\}, \quad g \circ f = \{(1,\alpha), (2,\alpha)\}$

② Onto

A function $f: A \to B$ is onto (or surjective) iff for all $y \in B$ there is an $x \in A$ such that $y = f(x)$.



Idea: Everything in B gets hit by something (or maybe many things) in A.



If $f: A \to B$ is not onto we can make it onto by replacing B by Rng(f).

es #1(g) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \sin(x)$ not onto because there is no x such that $\sin(x) = 2$.

(h) $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $f(x,y) = x - y$. onto? YES.

Pf: Let $\alpha \in \mathbb{R}$. Need to find $x, y \in \mathbb{R}$ such that $x - y = \alpha$. So take for example $x = \alpha, y = 0$. Then $x - y = \alpha - 0 = \alpha$.