Announcements

- Class will start at 5 pm on Wed July 8.
- Exam 5 will cover 3.2–3.4

Exam 4, #3.

If p not prime, result does not hold.

e.g. $6 \mid 9 \cdot 4$ but $6 \nmid 9$ and $6 \nmid 4$

$7 \mid 35 \Leftrightarrow 7 \mid 5 \cdot 7$

Suppose result holds for $k = 2, 3, 4, \dots, n-1$. Prove it holds for n. Suppose that p is prime and $p \mid x_1 x_2 \cdots x_n$. Let $y_1 = x_1 x_2 \cdots x_{n-1}$ and $y_2 = x_n$. Then $p \mid y_1 y_2$. Since result holds for $n=2$ then either $p \mid y_1$ or $p \mid y_2$. If $p \mid y_2$ then $p \mid x_n$ and we are done. If $p \mid y_1$ then this means $p \mid x_1 x_2 \cdots x_{n-1}$. Since result holds for $n-1$, $p \mid x_i$ for some $i = 1, 2, \dots, n-1$. So result holds for n.

3.2 7 b, c, 4c, e, 1a
3.4 10
3.3 6a, b

3.2 1 (a) $R=\{(1,2)\}$ on $A=\{1,2\}$

reflexive: $\forall x\in A\ (x,x)\in R$ NO

symmetric: $\forall x,y\in A$ If $(x,y)\in R$ then $(y,x)\in R$ NO

transitive: $\forall x,y,z\in A$ if $(x,y)\in R$ and $(y,z)\in R$ then $(x,z)\in R$

YES since antecedent always false.

4 (a) Reflexive: Show that for all $x\in\mathbb{R}$, xVx. Let $x\in\mathbb{R}$. Then xVx because $x=x$.

Symmetric: Show that for all $x,y\in\mathbb{R}$, xVy implies yVx. Suppose that xVy. Then either $x=y$ or $xy=1$. If $x=y$ then $y=x$ so yVx. If $xy=1$ then $yx=1$ so yVx in this case also. Hence yVx.

Transitive: Show that for all $x,y,z\in\mathbb{R}$, if xVy and yVz then xVz. Suppose xVy and yVz. If $x=y$ and $yz=1$ then $xz=1$ so xVz. If $x=y$ and $y=z$ then $x=z$ and xVz. If $xy=1$ and $y=z$ then $xz=1$ and xVz. If $xy=1$ and $yz=1$, then $xy=yz$ so $x=z$ as long as $y\neq 0$. But $y\neq 0$ since $xy=1$. Hence in all cases xVz.

(c) $3/V=\{3,\frac{1}{3}\}$ $\quad 0/V=\{0\}$

(e) $(x,y)\,T\,(a,b)$ iff $x^2+y^2=a^2+b^2$

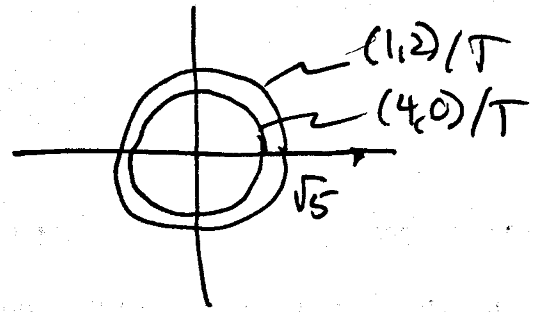
Reflexive: $(x,y)\,T\,(x,y)$ means $x^2+y^2=x^2+y^2$

Symm: $(x,y)\,T\,(a,b)\Rightarrow(a,b)\,T\,(x,y)$

Transitive: $(x,y)\,T\,(a,b)$ & $(a,b)\,T\,(c,d)\Rightarrow(x,y)\,T\,(c,d)$

$(1,2)/T =$

$\{(x,y) : x^2+y^2=5\}$



$(x,y)\,T\,(1,2) \iff x^2+y^2 = 1^2+2^2 = 5$

7 (b) $\equiv_m$ $\quad x \equiv_m y$ iff $m \mid (y-x)$

$\bar{x} = x/\equiv_m$ $\quad x \in \bar{x}$ means $x \equiv_m x$

means $m \mid 0$ which is true

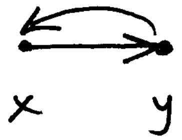
3.3 6 a, b

(a) R an equiv relu on $\mathbb{N}$ by
$n R m$ iff there exists $k \in \mathbb{N}$ such that $2^{k-1} \leq n, m < 2^k$

3.4 (10) $x_1 x_2 \leq y_1 y_2$ iff (i) $x_1 < y_1$, (ii) $x_1 = y_1$ and $x_2 \leq y_2$

3.5 Graphs.

We have seen directed graphs.

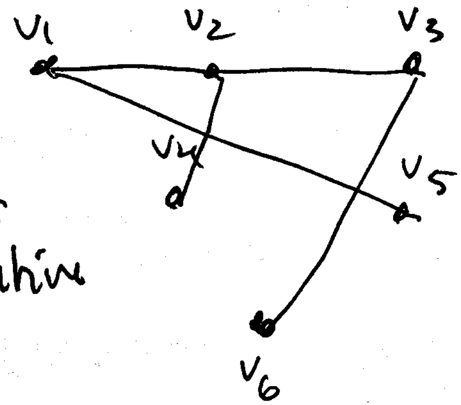


$x R y$

$y R x$ (if symmetric)

<u>Def</u>: A <u>graph</u> is a pair (V, E) where V is a nonempty set (the <u>vertices</u>) and E is a set of unordered pairs of distinct elements of V (called <u>edges</u>)

1) Graphs are represented by



not transitive

$V = \{v_1, v_2, v_3, \ldots, v_6\}$

$E = \{v_1v_2, v_1v_3, v_1v_5, v_2v_4, v_2v_3, v_3v_6\}$

Note: Don't distinguish v_1v_2 and v_2v_1.

Also never get v_3v_3 for example.

Think of graph as a relation on V that is (i) not reflexive, (ii) symmetric, (iii) not necessarily transitive.

Def: Let $G=(V,E)$ be a graph.

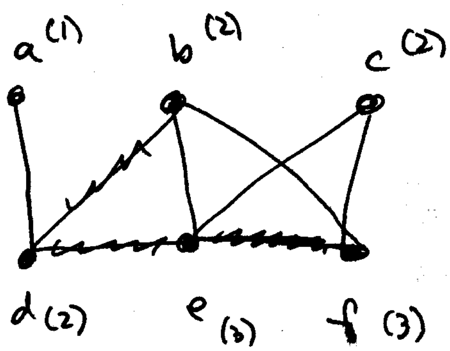
The order of G is $\overline{\overline{V}}$, ie the number of vertices

The size of G is $\overline{\overline{E}}$, the number of edges.

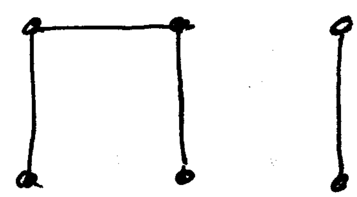
The degree of a vertex $u \in V$ is the number of edges incident with u, ie the number of edges uv where $v \in V$.

eg #2 (d)

$\sum \deg = 1+2+2+2+3+3 = 13$



(e)



works.

seems impossible

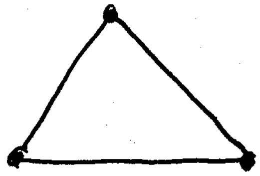
why impossible?

Thm: The sum of degrees of vertices in a graph must be even.

Pf: Since each edge is incident with exactly 2 vertices then counting the degree of each vertex counts each edge twice, so in fact

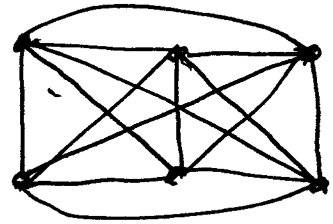
$$\sum_{\text{vertices}} \text{degrees} = 2 \cdot \#\text{edges}.$$

#3 (c)



impossible

(d)

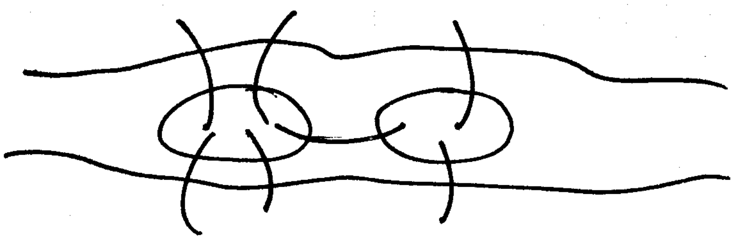


$1+2+3+\cdots+(n-1) = \frac{n(n-1)}{2}$

<u>Thm</u>: The size of a graph with order n can be at most $\frac{n(n-1)}{2}$ or $\binom{n}{2}$. Such a graph is called <u>complete</u>.

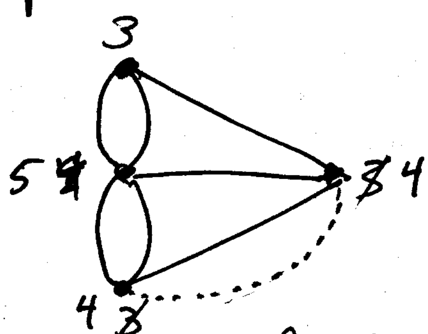
<u>Pf</u>: Each vertex can have degree at most $n-1$ and since there are n vertices, the sum of degrees is at most $n(n-1)$. This means there can be at most $\frac{n(n-1)}{2}$ edges. (<u>OR</u> each edge is a 2-element subset of V and there are $\binom{n}{2}$ of these and $\binom{n}{2} = \frac{n(n-1)}{2}$)

Königsburg bridges



Can you take a walk crossing each bridge exactly once? Euler: NO

① Represent as a graph



(This is called a multigraph since multiple edges between points is allowed)

② If there were such a walk (called an Euler walk) would need to be an even # of edges incident to each vertex except possibly 2 (start & finish)

Thm (Euler). If a graph (multigraph) G has more than 2 vertices of odd degree then it has no Euler walk.

More specifically

(1) If G has no vertex of odd degree it has at least one Euler circuit

(2) If G has 2 vertices of odd degree it has at least one Euler walk.

(3) If G has more than 2 odd vertices it has neither.

4.1 Functions as relations

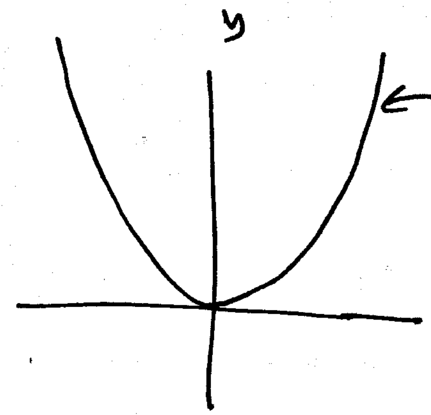
① Functions are a particular type of relation

Familiar notion of function on $\mathbb{R}$

(i) formula $y=f(x)$

(ii) graphs

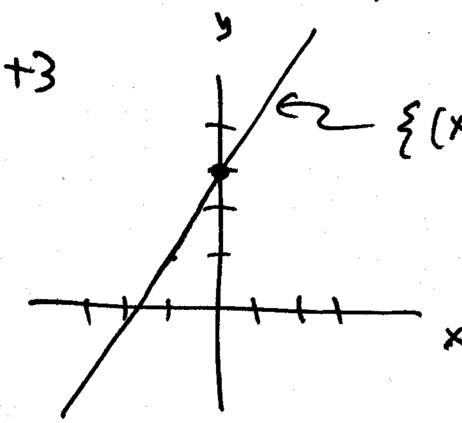
e.g. $y=x^2$



$\{(x,y) : y=x^2\}$

$f : \mathbb{R} \to \mathbb{R}$ is a function

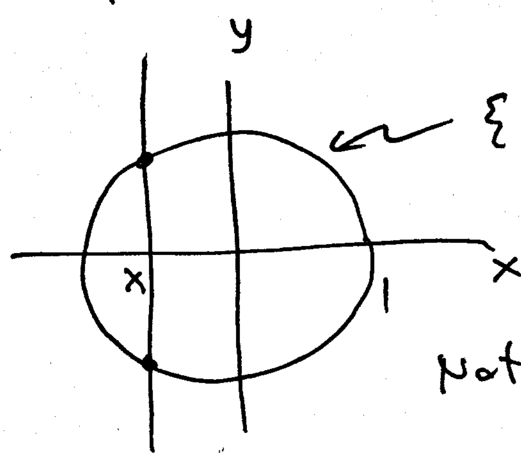
$y=2x+3$



$\{(x,y) : y=2x+3\}$

is a function

$x^2+y^2=1$

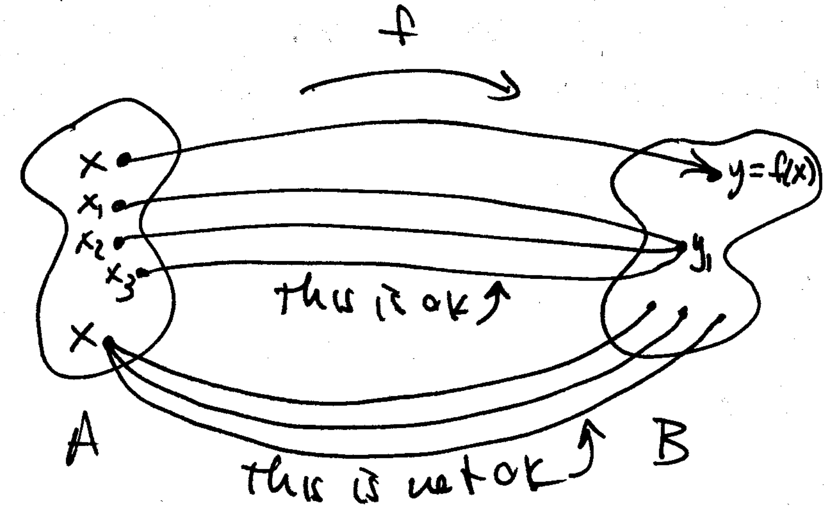


$\{(x,y) : x^2+y^2=1\}$

Not a function

Vertical line test: Any vertical line crosses graph of a function at at most one point.

This means that each x can be associated to at most one y.



(2) <u>Def</u>: A <u>function</u> from A to B is a relation f from A to B such that

(i) $\mathrm{Dom}(f) = A$

(ii) If $(x,y)\in f$ and $(x,z)\in f$ then $y=z$.

Write $f: A \to B$

(a) Part of the definition is that $\mathrm{Dom}(f)=A$.

(b) Informally a function is a "<u>rule</u> that assigns to each element of A one and only one (or exactly one) element of B."
In general the rule part is the <u>relation</u> f
Write $y=f(x)$ for $(x,y)\in f$. The <u>one and only one</u> part is given in part (ii) above.

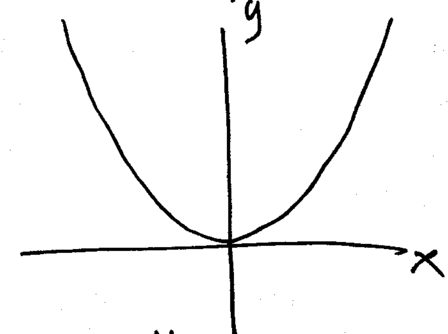
(c) Sometimes ~~the~~ f is given only by the rule and the domain A is implied

<u>eg</u> $f(x) = \sqrt{1-x^2}$ $\quad \text{Dom}(f) = [-1,1]$

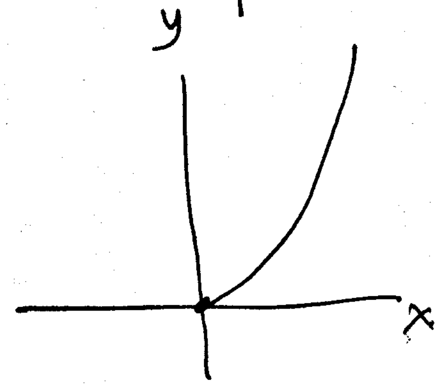
write $f : [-1,1] \to \mathbb{R}$

(d) The domain should be specified with the rule (but often is just implied). So for example

$f(x) = x^2, \ \underline{x \in \mathbb{R}}$



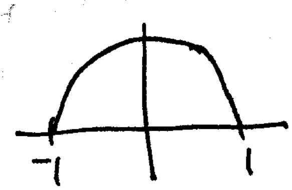
$f(x) = x^2, \ x \geq 0$



(e) The set B is the <u>co-domain</u> of f and is distinct from range of f, denoted Rng(f).

<u>e.g.</u> $f(x) = \sqrt{1-x^2}$ $\quad f : [-1,1] \to \mathbb{R}$ ← codomain

$\text{Rng}(f) = [0,1]$.



Examples :

#1 (f) $R_6 = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : y^2 = x\}$

Not a function : $(1,1)$ and $(1,-1) \in R_6$ but $1 \neq -1$.

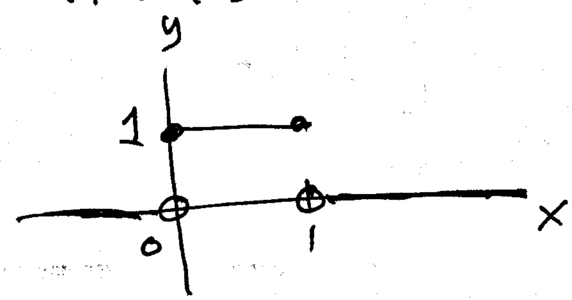
(h) $R_8 = \{(x,y) \in \mathbb{N} \times \mathbb{N} : y^2 = x\}$

is a function $R_8 : \mathbb{N} \to \mathbb{N}$

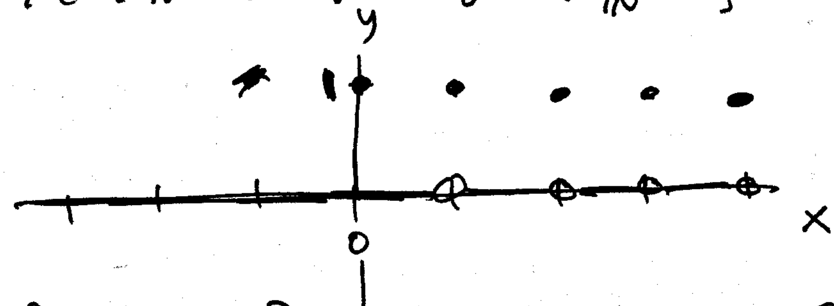
#3. Let $D \subseteq A$ then $\chi_D : A \to \mathbb{R}$ is

$$\chi_D(x) = \begin{cases} 1 & \text{if } x \in D \\ 0 & \text{if } x \notin D \end{cases}$$

$\chi_{[0,1]}(x)$



(f) $\{(x,y) \in \mathbb{R} \times \mathbb{R} : y = \chi_{\mathbb{N}}(x)\} = f$



$f : \mathbb{R} \to \mathbb{R}$ or $f : \mathbb{R} \to \{0,1\}$

#14. Canonical map $f: \mathbb{Z} \to \mathbb{Z}_6$

$$\mathbb{Z}_6 = \mathbb{Z}/\equiv_6 \qquad \mathbb{Z}_6 = \{[0], [1], \ldots, [5]\}$$

$$[0] = \{\ldots, -12, -6, 0, 6, 12, \ldots\} \qquad [1] = \{\ldots, -11, -5, 1, 7, 13, \ldots\}$$

$f: \mathbb{Z} \to \mathbb{Z}_6$ defined by

$$f(x) = x/\equiv_6$$

Using $\overline{3} = \cancel{3}\ 3/\equiv_6 = \{\ldots, -9, -3, 3, 9, 15, 21, \ldots\}$

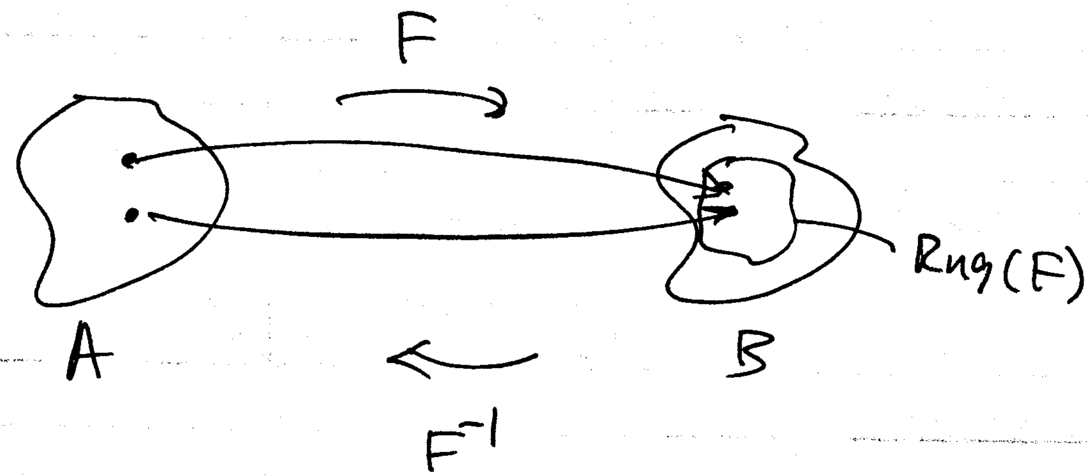
4.2 Construction of functions

① Inverses

If $F: A \to B$ is a function, the inverse relation F^{-1} is given by

$$F^{-1} = \{(x,y) : (y,x) \in F\}$$

- Sometimes F^{-1} is a function, sometimes not.
- If it is then $F^{-1}: \mathrm{Rng}(F) \to A = \mathrm{Dom}(F)$

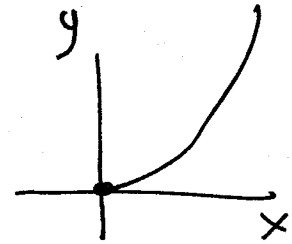


e.g. $f(x) = x^2 \quad f: \mathbb{R} \to \mathbb{R}$

$$f = \{(x,y) \in \mathbb{R}\times\mathbb{R} : y = x^2\}$$

$$f^{-1} = \{(x,y) \in \mathbb{R}\times\mathbb{R} : x = y^2\} = \{(x,y)\in \mathbb{R}\times\mathbb{R} : (y,x) \in f\}$$

Is f^{-1} a function? No. $(1,-1) \in f^{-1}$ and $(1,1) \in f^{-1}$.



eg $f(x) = x^2$ $\quad f: [0,\infty) \to \mathbb{R}$

$$f = \{(x,y) \in [0,\infty) \times \mathbb{R} : y = x^2\}$$

$$f^{-1} = \{(x,y) \in \mathbb{R} \times [0,\infty) : (y,x) \in f\}$$

$$= \{(x,y) \in \mathbb{R} \times [0,\infty) : x = y^2\}$$

f^{-1} is a function in this case, and we write

$f^{-1}: [0,\infty) \to \mathbb{R}$ or $f^{-1}: [0,\infty) \to [0,\infty)$

$$f^{-1}(x) = \sqrt{x}$$