Notation

Relation R on A

x/R = "x mod R" $= \{y \in A : xRy\}$

$A/R = \{x/R : x \in A\}$

eg #4(f) 3.2

How many elements in $\mathcal{P}(X)/R$? 7

$X = \{m, n, p, q, r, s\}$ $\quad ARB$ iff $\bar{\bar{A}} = \bar{\bar{B}}$

e.g. $\{m\}/R \in \mathcal{P}(X)/R \qquad \{n\}/R$

$\{\{m\}, \{n\}, \{p\}, \{q\}, \{r\}, \{s\}\}$

$\emptyset \in \mathcal{P}(X) \qquad \emptyset/R = \{\emptyset\}$

$\{m,n\}/R = \{\{m,n\}, \{m,p\}, \{m,q\}, \ldots, \{r,s\}\}$

How many here? $\binom{6}{2}$

3.4 Ordering Relations.

① Equivalence relations

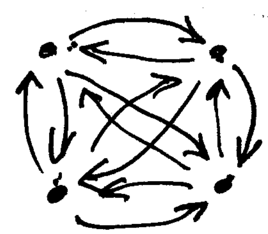
- generalizes the notion of "=" or "equivalence"
- divide the set into non-empty, disjoint equivalence classes (partition)

- Think about digraph representation:

reflexive $\Leftrightarrow$ loops on all pts.

symmetric $\Leftrightarrow$ all edges go both ways

transitive $\Leftrightarrow$ each piece of graph is "complete", i.e. all pts connected to all other pts.



(2) Partial order

- Generalize the relation $\leq$ on $\mathbb{R}$
- Properties: reflexive? $x \leq x \ \forall x$ <u>YES</u>

  symmetric? $x \leq y \Rightarrow y \leq x \ \forall x, y$ <u>NO</u>

  transitive? $x \leq y + y \leq z \Rightarrow x \leq z \ \forall x, y, z$ <u>YES</u>

- Example: "divides" on $\mathbb{Z}$

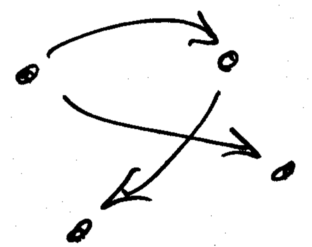
  $x R y$ iff $x | y$. reflexive, not symmetric, transitive.

- Example: Given a set A. Define R on $\mathcal{P}(A)$ by $A R B$ iff $A \subseteq B$. reflexive, not symm., transitive.

One more property of these examples:

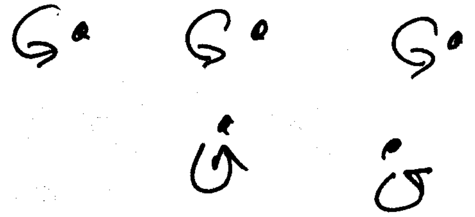
<u>Def</u>: A relation R on A is <u>anti-symmetric</u> if for all $x, y \in A$, if xRy and yRx then $x = y$

— If R is antisymmetric on finite set A, digraph looks like:

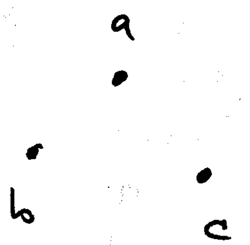


Never have:

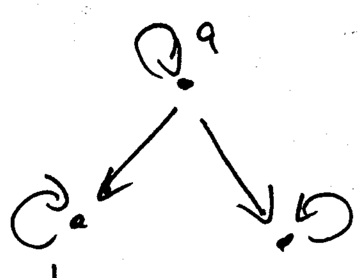
— If an equivalence relation were also antisymmetric what would it look like?



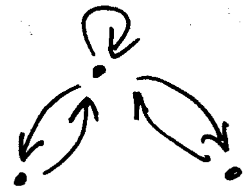
#3 (a)



(b)



(d)

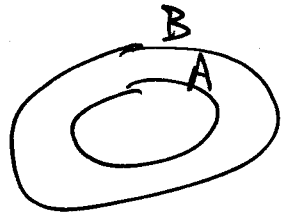


– In above relations (~~$\leq$ on $\mathbb{R}$~~, divides on $\mathbb{Z}$, $\subseteq$ on $\mathcal{P}(A)$) all elements ~~the~~ need not be comparable.

e.g. $6 \mid 18$ so 6 & 18 are comparable

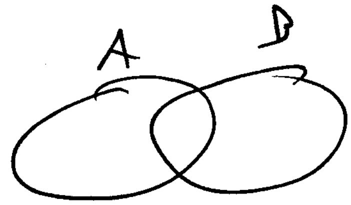
but 7 and 11 are not.

Sets



A & B comparable

$A \subseteq B$



A & B not comparable

$A \not\subseteq B$ and $B \not\subseteq A$.

$\leq$ on $\mathbb{R}$, all elements are comparable.

$\forall x, y \in \mathbb{R}$ either $x \leq y$ or $y \leq x$

<u>Def</u>: A relation R on A is a <u>partial order</u> if R is reflexive, anti-symmetric and transitive. R is a <u>linear order</u> (or <u>total order</u>) if it is a partial order and for all $x, y \in A$ either xRy or yRx.

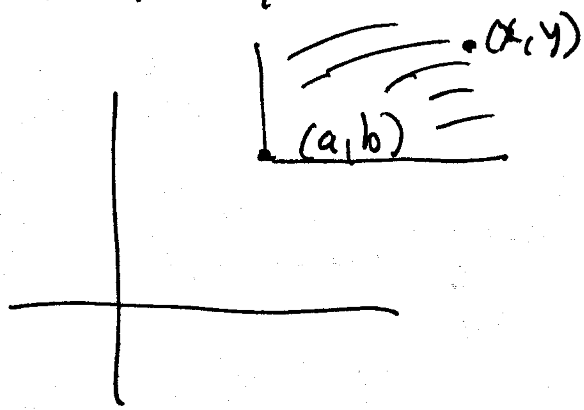
#7) <u>Pf</u>: (1) Since $a = 2^0 a$ for all $a \in \mathbb{N}$, R is reflexive.

(2) Suppose $a = 2^k b$ and $b = 2^n a$ for some integers k and $n \geq 0$. This means $a = 2^k b = 2^k 2^n a = 2^{n+k} a$

Therefore $n + k = 0$ or $n = -k$. ~~Thus~~ However $\cancel{a = 2^k b}$ $\cancel{b = 2^n a}$ since $n \geq 0$ and $k \geq 0$ we must have $n = k = 0$. Thus $a = b$.

(3) Suppose $a = 2^k b$ and $b = 2^n c$ for some $a, b, c \in \mathbb{N}$, $k, n \geq 0$ integers. Then $a = 2^k b = 2^k 2^n c = 2^{n+k} c$. Since $n + k \geq 0$ is an integer $a R c$.

#8)



$(a,b) \, R \, (x,y)$ iff $a \leq x$ and $b \leq y$

③ Hasse Diagram

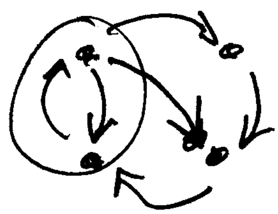
Let R be a partial order on A (finite set)

What do we know about digraph of R?

1) Always has

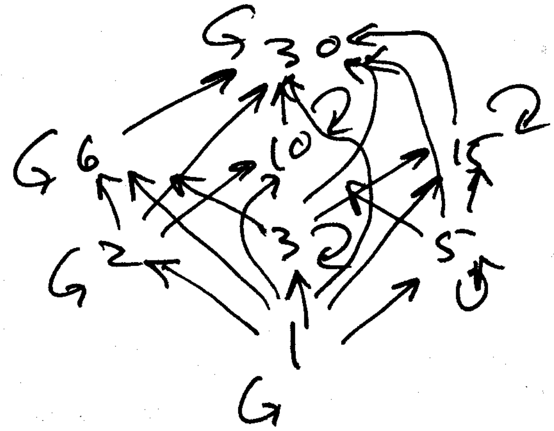
2) Never have 

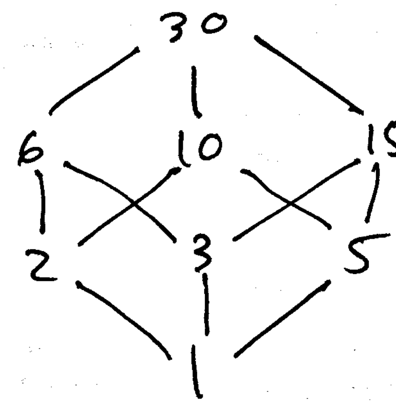
3) Never have a "loop"



e.g. $M = \{1,2,3,5,6,10,15,30\}$ the divisors of 30

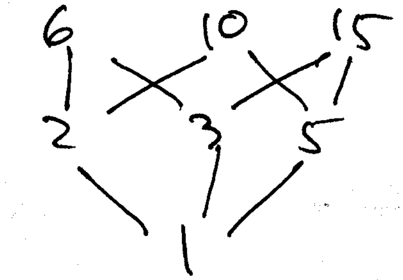
Let D be "divides" on M.





Simplify by
(1) eliminate
(2) get rid of arrowheads
i.e. $\rightarrow$ becomes —
(3) elimate all edges that come from transitivity

Note: 1 is the "smallest" element of M and 30 is the "largest" e.g. S'pse we just did D on $\{1,2,3,5,6,10,15\}$



No largest element.

#13 (a) $A = \{t_1, t_2, t_3, \ldots, t_9\}$
and R on A defined by "comes before" is a partial order.