
Given that the eigenvalues of the matrix A are $\lambda = 0$ and $\lambda = 1$, find all of the eigenvectors.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 0 & 0 \end{pmatrix}$$

Solution: Finding the eigenvectors corresponding to $\lambda = 0$, we solve $A\mathbf{x} = \mathbf{0}$, that is,

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is simple enough to be solved directly without using Gaussian elimination, since the first equation is $x_1 = 0$, is equivalent to the third, $3x_1 = 0$, and we can take $x_3 = t$ as the free variable. The second equation gives $x_2 = 2t$, and so the eigenvectors all have the form

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

Finding the eigenvectors corresponding to $\lambda = 1$, we solve $(A - I)\mathbf{x} = \mathbf{0}$, that is,

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Again this is simple enough to be solved directly without using Gaussian elimination. The last two equations reduce to the 2×2 system $2x_1 - 2x_3 = 0$; $3x_1 - x_3 = 0$ which solves to $x_1 = x_3 = 0$. We can take $x_2 = t$ to be a free variable, and hence all eigenvectors have the form

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Given that the eigenvalues of the matrix A are $\lambda = 0$ and $\lambda = 4$, find all of the eigenvectors.

$$A = \begin{pmatrix} 5 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution: Finding the eigenvectors corresponding to $\lambda = 0$, we solve $A\mathbf{x} = \mathbf{0}$, that is,

$$\begin{pmatrix} 5 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is simple enough to be solved directly without using Gaussian elimination. Taking $x_3 = t$ to be the free variable leaves us with the 2×2 system $5x_1 - x_2 = 0$; $x_1 + 3x_2 = 0$ which solves to $x_1 = x_2 = 0$. Hence the eigenvectors all have the form

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Finding the eigenvectors corresponding to $\lambda = 4$, we solve $(A - 4I)\mathbf{x} = \mathbf{0}$, that is,

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Again this is simple enough to be solved directly without using Gaussian elimination. The last equation is $-4x_3 = 0$ giving $x_3 = 0$. This leaves us with the single equation $x_1 - x_2 = 0$. Taking $x_2 = t$ to be a free variable leads to $x_1 = t$. Hence all eigenvectors have the form

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Given that two eigenvalues of the matrix A are $\lambda = 0$ and $\lambda = 1$, find all of the corresponding eigenvectors.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution: Finding the eigenvectors corresponding to $\lambda = 0$, we solve $A\mathbf{x} = \mathbf{0}$, that is,

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is simple enough to be solved directly without using Gaussian elimination. The first equation gives $x_1 = 0$, and the remaining equations reduce to the single equation $x_2 + x_3 = 0$. Taking $x_3 = t$ as the free variable, we conclude that $x_2 = -t$ and hence that all eigenvectors have the form

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Finding the eigenvectors corresponding to $\lambda = 1$, we solve $(A - I)\mathbf{x} = \mathbf{0}$, that is,

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Again this is simple enough to be solved directly without using Gaussian elimination. Taking $x_1 = t$ to be a free variable, the second equation gives $x_3 = -t$, and the third gives $x_2 = -t$. If we had started with any other variable as free we would arrive at the same conclusion. Hence all eigenvectors have the form

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$