MATH 214 – QUIZ 13 – SOLUTIONS

Use the Laplace transform (and the table below) to solve the initial value problem

$$y'' + 3y' + 2y = u_{10}(t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: Taking the Laplace transform of both sides gives

$$\begin{aligned}
\mathcal{L}\{y'' + 3y' + 2y\} &= \mathcal{L}\{u_{10}(t)\} \\
\mathcal{L}\{y''\} + 3\,\mathcal{L}\{y'\} + 2\,\mathcal{L}\{y\} &= \frac{e^{-10s}}{s} \\
s^2\,\mathcal{L}\{y\} - s\,y(0) - y'(0) + 3(s\,\mathcal{L}\{y\} - y(0)) + 2\,\mathcal{L}\{y\} &= \frac{e^{-10s}}{s} \\
(s^2 + 3s + 2)\mathcal{L}\{y\} &= \frac{e^{-10s}}{s}
\end{aligned}$$

so that

$$\mathcal{L}\{y\} = e^{-10s}\,\frac{1}{s(s+2)(s+1)}.$$

Expanding this last term in partial fractions gives

$$\frac{1}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1} = \frac{A(s+1)(s+2) + Bs(s+1) + Cs(s+2)}{(s+2)(s+1)}$$

so that

$$A(s+1)(s+2) + Bs(s+1) + Cs(s+2) = 1.$$

Plugging in $s = 0$ gives $A = 1/2$, plugging in $s = -2$ gives $B = 1/2$ and plugging in $s = -1$ gives $C = -1$. Therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{1}{(s+2)} - \frac{1}{(s+1)}\right\} = \frac{1}{2} + \frac{1}{2}\,e^{-2t} - e^{-t}.$$

Finally,

$$y = u_{10}(t)\left(\frac{1}{2} + \frac{1}{2}\,e^{-2(t-10)} - e^{-(t-10)}\right).$$

Use the Laplace transform (and the table below) to solve the initial value problem

$$y'' - 4y' + 4y = \delta(t-5) + u_{10}(t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: Taking the Laplace transform of both sides gives

$$\begin{aligned}
\mathcal{L}\{y'' - 4y' + 4y\} &= \mathcal{L}\{\delta(t-5)\} + \mathcal{L}\{u_{10}(t)\} \\
\mathcal{L}\{y''\} - 4\,\mathcal{L}\{y'\} + 4\,\mathcal{L}\{y\} &= e^{-5t} + \frac{e^{-10s}}{s} \\
s^2\,\mathcal{L}\{y\} - s\,y(0) - y'(0) - 4(s\,\mathcal{L}\{y\} - y(0)) + 4\,\mathcal{L}\{y\} &= e^{-5t} + \frac{e^{-10s}}{s} \\
(s^2 - 4s + 4)\mathcal{L}\{y\} &= e^{-5t} + \frac{e^{-10s}}{s}
\end{aligned}$$

so that

$$\mathcal{L}\{y\} = e^{-5t}\,\frac{1}{(s-2)^2} + e^{-10t}\,\frac{1}{s(s-2)^2}.$$

Expanding the second term in partial fractions gives

$$\frac{1}{s(s-2)^2} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2} = \frac{A(s-2)^2 + Bs(s-2) + Cs}{s(s-2)^2}$$

so that

$$A(s-2)^2 + Bs(s-2) + Cs = 1.$$

Plugging in $s = 0$ gives $A = 1/4$, plugging in $s = 2$ gives $C = 1/2$ and taking a derivative gives the equation $2A(s-2) + Bs + B(s-2) + C = 0$ and plugging $s = 2$ into this equation gives $B = -1/4$. Therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{1}{(s-2)} + \frac{1}{2}\frac{1}{(s-2)^2}\right\} = \frac{1}{4} - \frac{1}{4}\,e^{2t} + \frac{1}{2}\,t\,e^{2t}.$$

Since also

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} = t\,e^{2t},$$

we finally arrive at

$$y = u_5(t)\left((t-5)\,e^{2(t-5)}\right) + u_{10}(t)\left(\frac{1}{4} - \frac{1}{4}\,e^{2(t-10)} + \frac{1}{2}\,(t-10)\,e^{2(t-10)}\right).$$

Use the Laplace transform (and the table below) to solve the initial value problem

$$y'' - y' - 6y = \delta(t-5) - \delta(t-10), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: Taking the Laplace transform of both sides gives

$$\begin{aligned}
\mathcal{L}\{y'' - y' - 6y\} &= \mathcal{L}\{\delta(t-5)\} - \mathcal{L}\{\delta(t-10)\} \\
\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\,\mathcal{L}\{y\} &= e^{-5t} - e^{-10t} \\
s^2\,\mathcal{L}\{y\} - s\,y(0) - y'(0) - (s\,\mathcal{L}\{y\} - y(0)) - 6\,\mathcal{L}\{y\} &= e^{-5t} - e^{-10t} \\
(s^2 - s - 6)\mathcal{L}\{y\} &= e^{-5t} - e^{-10t}
\end{aligned}$$

so that

$$\mathcal{L}\{y\} = (e^{-5t} - e^{-10t})\,\frac{1}{(s+2)(s-3)}.$$

Expanding this last term in partial fractions gives

$$\frac{1}{(s+2)(s-3)} = \frac{A}{s+2} + \frac{B}{s-3} = \frac{A(s-3) + B(s+2)}{(s+2)(s-3)}$$

so that

$$A(s-3) + B(s+2) = 1.$$

Plugging in $s = -2$ gives $A = -1/5$ and plugging in $s = 3$ gives $B = 1/5$. Therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{5}\,\frac{1}{(s-3)} - \frac{1}{5}\,\frac{1}{(s+2)}\right\} = \frac{1}{5}\,e^{3t} - \frac{1}{5}\,e^{-2t}.$$

Finally,

$$y = \frac{1}{5}\,u_5(t)\left(e^{3(t-5)} - e^{-2(t-5)}\right) + \frac{1}{5}\,u_{10}(t)\left(e^{3(t-10)} - e^{-2(t-10)}\right).$$