MATH 214 – QUIZ 12 – SOLUTIONS

Use the Laplace transform (and the table below) to solve the initial value problem

$$y'' + 3y' + 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: Taking the Laplace transform of both sides gives

$$\begin{aligned} \mathcal{L}\{y'' + 3y' + 2y\} &= 0 \\ \mathcal{L}\{y''\} + 3\,\mathcal{L}\{y'\} + 2\,\mathcal{L}\{y\} &= 0 \\ s^2\,\mathcal{L}\{y\} - s\,y(0) - y'(0) + 3(s\,\mathcal{L}\{y\} - y(0)) + 2\,\mathcal{L}\{y\} &= 0 \\ (s^2 + 3s + 2)\mathcal{L}\{y\} - s - 3 &= 0 \end{aligned}$$

so that

$$\mathcal{L}\{y\} = \frac{s+3}{s^2 + 3s + 2}.$$

Expanding this last term in partial fractions gives

$$\frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1} = \frac{A(s+1) + B(s+2)}{(s+2)(s+1)}$$

so that

$$A(s+1) + B(s+2) = s + 3.$$

Plugging in $s = -2$ gives $A = -1$ and plugging in $s = -1$ gives $B = 2$. Therefore

$$y = \mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{1}{s+2}\right\} = 2\,e^{-t} - \,e^{-2t}.$$

Use the Laplace transform (and the table below) to solve the initial value problem

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Taking the Laplace transform of both sides gives

$$\begin{aligned} \mathcal{L}\{y'' - 4y' + 4y\} &= 0 \\ \mathcal{L}\{y''\} - 4\,\mathcal{L}\{y'\} + 4\,\mathcal{L}\{y\} &= 0 \\ s^2\,\mathcal{L}\{y\} - s\,y(0) - y'(0) - 4(s\,\mathcal{L}\{y\} - y(0)) + 4\,\mathcal{L}\{y\} &= 0 \\ (s^2 - 4s + 4)\mathcal{L}\{y\} - s + 3 &= 0 \end{aligned}$$

so that

$$\mathcal{L}\{y\} = \frac{s-3}{s^2-4s+4}.$$

Expanding this last term in partial fractions gives

$$\frac{s-3}{s^2-4s+4} = \frac{s-3}{(s-2)^2} = \frac{A}{s-2} + \frac{B}{(s-2)^2} = \frac{A(s-2)+B}{(s-2)^2}$$

so that

$$A(s-2)+B = s-3.$$

Plugging in $s=2$ gives $B=-1$ and taking a derivative gives immediately $A=1$. Therefore

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s-2} - \frac{1}{(s-2)^2}\right\} = e^{2t} - t\,e^{2t}.$$

Use the Laplace transform (and the table below) to solve the initial value problem

$$y'' - y' - 6y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Taking the Laplace transform of both sides gives

$$\begin{aligned}
\mathcal{L}\{y'' - y' - 6y\} &= 0\\
\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\,\mathcal{L}\{y\} &= 0\\
s^2\,\mathcal{L}\{y\} - s\,y(0) - y'(0) - (s\,\mathcal{L}\{y\} - y(0)) - 6\,\mathcal{L}\{y\} &= 0\\
(s^2 - s - 6)\mathcal{L}\{y\} - s &= 0
\end{aligned}$$

so that

$$\mathcal{L}\{y\} = \frac{s}{s^2-s-6}.$$

Expanding this last term in partial fractions gives

$$\frac{s-5}{s^2-s-6} = \frac{s}{(s+2)(s-3))} = \frac{A}{s+2} + \frac{B}{(s-3)} = \frac{A(s-3)+B}{(s+2)}$$

so that

$$A(s-3) + B(s+2) = s.$$

Plugging in $s=-2$ gives $A = 2/5$ and plugging in $s=3$ gives $B = 3/5$. Therefore

$$y = \mathcal{L}^{-1}\left\{\frac{2}{5}\,\frac{1}{s+2} + \frac{3}{5}\,\frac{1}{(s-3)}\right\} = \frac{2}{5}\,e^{-2t} + \frac{3}{5}\,e^{3t}.$$