
Find the solution to the initial value problem.

$$y''' - 4y'' + 4y' = 0, \qquad y(0) = 1, \qquad y'(0) = -1, \qquad y''(0) = 0.$$

Solution: The characteristic equation of this ODE is $r^3 - 4r^2 + 4r = 0$, which factors as $r(r-2)^2 = 0$ and hence has solutions $r_1 = 0$, $r_2 = 2$, where r_2 is a double root. Therefore the the general solution is given by

$$y(t) = c_1 + c_2\, e^{2t} + c_3\, t\, e^{2t}.$$

We have then $y'(t) = 2c_2\, e^{2t} + c_3\,(2t\, e^{2t} + e^{2t})$ and $y''(t) = 4c_2\, e^{2t} + c_3\,(4t\, e^{2t} + 4e^{2t})$. Plugging in the initial conditions gives the system of equations

$$\begin{aligned} 1 &= c_1 + c_2 \\ -1 &= 2c_2 + c_3 \\ 0 &= 4c_2 + 4c_3. \end{aligned}$$

Subtracting twice the second equation to the third gives $c_3 = 1$, and plugging this into the last equation gives $c_2 = -1$, and plugging this into the first equation gives $c_1 = 2$. Hence the final solution is

$$y(t) = 2 - e^{2t} + t\, e^{2t}.$$

Find the solution to the initial value problem.

$$y''' + y' = 0, \qquad y(0) = 0, \qquad y'(0) = 1, \qquad y''(0) = 2.$$

Solution: The characteristic equation of this ODE is $r^3 + r = 0$, which factors as $r(r^2+1) = 0$ and hence has solutions $r_1 = 0$, $r_2 = i$, $r_3 = -i$. Therefore the the general solution is given by

$$y(t) = c_1 + c_2\, \cos(t) + c_3\, \sin(t).$$

We have then $y'(t) = -c_2\, \sin(t) + c_3\, \cos(t)$ and $y''(t) = -c_2\, \cos(t) - c_3\, \sin(t)$. Plugging in the initial conditions gives the system of equations

$$\begin{aligned} 0 &= c_1 + c_2 \\ 1 &= c_3 \\ 2 &= -c_2 \end{aligned}$$

which leads to $c_1 = 2$, $c_2 = -2$, and $c_3 = 1$. Hence the final solution is

$$y(t) = 2 - 2\cos(t) + \sin(t).$$

Find the solution to the initial value problem.

$$y''' + 5y'' + 6y' = 0, \qquad y(0) = 0, \qquad y'(0) = 0, \qquad y''(0) = 1.$$

Solution: The characteristic equation of this ODE is $r^3 + 5r^2 + 6r = 0$, which factors as $r(r+3)(r+2) = 0$ and hence has solutions $r_1 = 0$, $r_2 = -3$, $r_3 = -2$. Therefore the the general solution is given by

$$y(t) = c_1 + c_2\, e^{-3t} + c_3\, e^{-2t}.$$

We have then $y'(t) = -3c_2\, e^{-3t} - 2c_3\, e^{-2t}$ and $y''(t) = 9c_2\, e^{-3t} + 4c_3\, e^{-2t}$. Plugging in the initial conditions gives the system of equations

$$\begin{aligned} 0 &= c_1 + c_2 + c_3 \\ 0 &= -3c_2 - 2c_3 \\ 1 &= 9c_2 + 4c_3. \end{aligned}$$

Adding 3 times the second equation to the third gives $c_3 = -1/2$, and plugging this into the second equation gives $c_2 = 1/3$, and plugging this into the first equation gives $c_1 = 1/6$. Hence the final solution is

$$y(t) = \frac{1}{6} + \frac{1}{3}\, e^{-3t} - \frac{1}{2}\, e^{-2t}.$$