MATH 214 – QUIZ 9 – SOLUTIONS

Use the method of variation of parameters to find a particular solution to the differential equation.

$$y'' + 2y' + y = 2\,e^{-t}.$$

Solution: The characteristic polynomial of this equation is $r^2+2r+1=(r+1)^2$ which has a double root at $r=-1$. Hence a fundamental set of solutions to the homogeneous equation is $\{e^{-t},\, t\,e^{-t}\}$.

We look for a particular solution of the form $Y(t)=u_1(t)\,e^{-t}+u_2(t)\,t\,e^{-t}$. This leads to the system of equations

$$\begin{aligned} u_1'(t)\,e^{-t}+u_2'(t)\,t\,e^{-t} &= 0 \\ -u_1'(t)\,e^{-t}+u_2'(t)(-t\,e^{-t}+e^{-t}) &= 2\,e^{-t}. \end{aligned}$$

Adding the equations together and factoring out the e^{-t} gives $u_2'(t)=2$, so that $u_2(t)=2t$. Plugging this solution into the first equation gives $u_1'(t)=-2t$, so that $u_1(t)=-t^2$. Therefore, a particular solution is given by

$$Y(t)=-t^2\,e^{-t}+2t^2\,e^{-t}=t^2\,e^{-t}.$$

Use the method of variation of parameters to find a particular solution to the differential equation.

$$y'' - 2y' + y = 4\,e^{-3t}.$$

Solution: The characteristic polynomial of this equation is $r^2-2r+1=(r-1)^2$ which has a double root at $r=1$. Hence a fundamental set of solutions to the homogeneous equation is $\{e^{t},\, t\,e^{t}\}$.

We look for a particular solution of the form $Y(t)=u_1(t)\,e^{t}+u_2(t)\,t\,e^{t}$. This leads to the system of equations

$$\begin{aligned} u_1'(t)\,e^{t}+u_2'(t)\,t\,e^{t} &= 0 \\ u_1'(t)\,e^{t}+u_2'(t)(t\,e^{t}+e^{t}) &= 4\,e^{-3t}. \end{aligned}$$

Subtracting the second equation from the first gives $u_2'(t) = 4e^{-4t}$, so that $u_2(t) = -e^{-4t}$. Plugging this solution into the first equation gives $u_1'(t) = -4t\,e^{-4t}$, so that $u_1(t) = t\,e^{-4t} + (1/4)\,e^{-4t}$. Therefore, a particular solution is given by

$$Y(t) = t\,e^{-3t} + \frac{1}{4}\,e^{-3t} - t\,e^{-3t} = \frac{1}{4}\,e^{-3t}.$$

Use the method of variation of parameters to find a particular solution to the differential equation.

$$y'' + 2y' = 4\,e^{3t}.$$

Solution: The characteristic polynomial of this equation is $r^2+2r = r(r+2)$ which has roots at $r = 0$ and $r = -2$. Hence a fundamental set of solutions to the homogeneous equation is $\{1,\, e^{-2t}\}$.

We look for a particular solution of the form $Y(t) = u_1(t) + u_2(t)\,e^{-2t}$. This leads to the system of equations

$$\begin{aligned} u_1'(t) + u_2'(t)\,e^{-2t} &= 0 \\ -2u_2'(t)\,e^{-2t} &= 4\,e^{3t}. \end{aligned}$$

The second equation gives immediately $u_2'(t) = -2\,e^{5t}$, so that $u_2(t) = -(2/5)e^{5t}$. Plugging this solution into the first equation gives $u_1'(t) = 2\,e^{3t}$, so that $u_1(t) = (2/3)\,e^{3t}$. Therefore, a particular solution is given by

$$Y(t) = \frac{2}{3}\,e^{3t} - \frac{2}{5}\,e^{3t} = \frac{4}{15}\,e^{3t}.$$