MATH 214 – QUIZ 2 – SOLUTIONS

Consider an electric circuit containing a capacitor, resistor, and battery. The charge $Q(t)$ on the capacitor satisfies the equation $\dfrac{dQ}{dt} + \dfrac{Q}{C} = V$, where C the capacitance, and V is the constant voltage supplied by the battery.

(a) (3 pts.) If $Q(0) = 0$, find $Q(t)$ for any time t (that is, solve the initial value problem).

(b) (2 pts.) Find the limiting value that $Q(t)$ approaches as $t \to \infty$.

Solution: (a) The equation is separable and can be rewritten $\dfrac{dQ}{dt} = V - \dfrac{Q}{C} = \dfrac{CV - Q}{C}$ which leads to

$$\begin{aligned} \frac{dQ}{dt} &= \frac{CV - Q}{C} \\ \frac{dQ}{CV - Q} &= \frac{dt}{C} \\ -\ln|CV - Q| &= \frac{t}{C} + k \\ |CV - Q| &= e^k e^{-t/C} \\ CV - Q &= c\, e^{-t/C} \\ Q &= CV - c\, e^{t/C} \end{aligned}$$

$Q(0) = 0$ leads to $c = 1$ so the final solution is $Q = CV(1 - e^{-t/C})$.

(b) As $t \to \infty$, $e^{-t/C} \to 0$ so that $Q(t) \to CV$ as can be seen from the above solution. It can also be seen from the original equation which has an equilibrium solution of $Q(t) = CV$. By looking at the direction field associated with this equation, it follows that CV is attracting and hence that all solutions converge to CV in the limit.

Consider an electric circuit containing a capacitor, resistor, and battery. The charge $Q(t)$ on the capacitor satisfies the equation $\dfrac{dQ}{dt} + \dfrac{Q}{C} = V$, where C the capacitance, and V is the constant voltage supplied by the battery.

(a) (3 pts.) Suppose that $Q(0) = Q_0 > 0$ (that is, the capacitor has charge Q_0 when $t = 0$), and that the battery is removed from the system and the circuit closed again (that is, we have set $V = 0$ in the above equation). Find $Q(t)$ for any time t (that is, solve the initial value problem).

(b) (2 pts.) Assuming the situation described in part (a), find the time T at which the charge on the capacitor is $Q_0/2$.

Solution: (a) The equation we are solving is $\dfrac{dQ}{dt} + \dfrac{Q}{C} = 0$ or $\dfrac{dQ}{dt} = -\dfrac{Q}{C}$ which leads to

$$\begin{aligned}
\frac{dQ}{dt} &= -\frac{Q}{C} \\
\frac{dQ}{Q} &= -\frac{dt}{C} \\
\ln|Q| &= -\frac{t}{C} + k \\
|Q| &= e^k e^{-t/C} \\
Q &= c\, e^{-t/C}
\end{aligned}$$

$Q(0) = Q_0$ leads to $c = Q_0$ so the final solution is $Q = Q_0\, e^{-t/C}$.

(b) We must solve $Q_0/2 = Q_0\, e^{-T/C}$ which leads to a solution of $T = C\ln(2)$.

Consider an electric circuit containing a capacitor, resistor, and battery. Suppose that the charge $Q(t)$ on the capacitor satisfies the equation $\dfrac{dQ}{dt} + \dfrac{Q}{C} = V$, where C the capacitance, and V is the constant voltage supplied by the battery.

(a) (3 pts.) Suppose that $C = .10$ farad, and $V = 10$ volts. If $Q(0) = 0$, find $Q(t)$ for any time t (that is, solve the initial value problem).

(b) (2 pts.) Find the time T required for the charge Q on the capacitor to reach .50 couloumb.

Solution: (a) The equation we are solving is $\dfrac{dQ}{dt} + \dfrac{Q}{.1} = 10$ or $\dfrac{dQ}{dt} = 10 - 10Q$. This equation is separable and solving it leads to

$$\begin{aligned}
\frac{dQ}{dt} &= 10(1-Q) \\
\frac{dQ}{1-Q} &= 10dt \\
-\ln|1-Q| &= 10t + k \\
|1-Q| &= e^k e^{-10t} \\
Q &= 1 - c\, e^{-10t}
\end{aligned}$$

$Q(0) = 0$ leads to $c = 1$ so the final solution is $Q = 1 -\ e^{-10t}$.

(b) We must solve $.50 = 1 -\ e^{-10T}$ which leads to a solution of $T = \ln(2)/10$.