MATH 214 – EXAM 2 VERSION 2 – SOLUTIONS

1. (10 pts.) Find the solution of the initial value problem

$$y'' + y' - 2y = 4t, \qquad y(0) = 0, \qquad y'(0) = 1.$$

Solution:

First we need to find the general solution to the homogeneous system. The characteristic equation for this ODE is $r^2 + r - 2 = 0$ which factors to $(r-1)(r+2) = 0$ giving a solution of $r = 1$ and $r = -2$. Hence the general solution is

$$y(t) = c_1\, e^t + c_2\, e^{-2t}.$$

We will use the method of undetermined coefficients to find a particular solution, of the form $Y(t) = A\,t + B$. This gives $Y'(t) = A$ and $Y''(t) = 0$. Plugging into the original equation gives

$$A - 2At - 2B = 4t$$

which solves to $A = -2$, $B = -1$. Hence the general solution is

$$y(t) = c_1\, e^t + c_2\, e^{-2t} - 2t - 1.$$

Plugging in the initial conditions leads to the system

$$\begin{aligned} 0 &= c_1 + c_2 - 1 \\ 1 &= c_1 - 2c_2 - 2 \end{aligned}$$

yielding a solution of $c_1 = 5/3$ and $c_2 = -2/3$. Hence the final solution is

$$y(t) = \frac{5}{3}\, e^t - \frac{2}{3}\, e^{-2t} - 2t - 1.$$

2. (10 pts.) Use the method of variation of parameters to find a particular solution to the differential equation

$$4y'' - 4y' + y = 16\, e^{t/2}.$$

Solution:

First we need to find the general solution to the homogeneous system. The characteristic equation for this ODE is $4r^2 - 4r + 1 = 0$ which factors to $(2r-1)^2 = 0$ giving solutions of $r = 1/2$ of multiplicity 2. Hence the general solution is

$$y(t) = c_1\, e^{t/2} + c_2\, t\, e^{t/2}.$$

We will use the method of variation of parameters to find a particular solution, of the form $Y(t) = u_1(t)\, e^{t/2} + u_2(t)\, t\, e^{t/2}$. The variation of parameters formula gives the system

$$\begin{aligned} u_1'(t)\, e^{t/2} + u_2'(t)\, t\, e^{t/2} &= 0 \\ \frac{1}{2}u_1'(t)\, e^{t/2} + u_2'(t)\left(\frac{t}{2}\, e^{t/2} + e^{t/2}\right) &= 4\, e^{t/2}. \end{aligned}$$

Note that the right side of the last equation comes from putting the original equation into standard form, $y'' - y + (1/4)y = 4\, e^{t/2}$. Subtracting twice the second equation from the first gives $-2u_2'(t)\, e^{t/2} = -8\, e^{t/2}$ which solve to $u_2'(t) = 4$ or $u_2(t) = 4t$. Plugging into the first equation gives $u_1'(t) = -4t$ or $u_1 = -2t^2$. Hence a particular solution is given by

$$Y(t) = -2t^2\, e^{t/2} + (4t)(t\, e^{t/2}) = 2t^2\, e^{t/2}.$$

3. (10 pts.) Find the general solution to the differential equation

$$y''' - y' = 2\, \sin(t).$$

Solution:

First we need to find the general solution to the homogeneous system. The characteristic equation for this ODE is $r^3 - r = 0$ which factors to $r(r^2 - 1) = r(r-1)(r+1) = 0$ giving solutions of $r = 0,\ 1,\ -1$. Hence the general solution is

$$y(t) = c_1 + c_2\, e^t + c_3\, e^{-t} + Y(t)$$

where $Y(t)$ is a particular solution. We will find $Y(t)$ by the method of undetermined coefficients. If $Y(t) = A\, \sin(t) + B\, \cos(t)$ then $Y'(t) = A\, \cos(t) - B\, \sin(t)$, $Y''(t) = -A\, \sin(t) - B\, \cos(t)$, and $Y'''(t) = -A\, \cos(t) + B\, \sin(t)$. Plugging into the original equation gives

$$-A\, \cos(t) + B\, \sin(t) - A\, \cos(t) + B\, \sin(t) = -2A\, \cos(t) + 2B\, \sin(t) = 2\, \sin(t)$$

which gives $A = 0$ and $B = 1$. Hence the general solution to the equation is

$$y(t) = c_1 + c_2\, e^t + c_3\, e^{-t} + \cos(t).$$

4. (10 pts.) Find the general solution to the homogeneous differential equation

$$y^{(4)} + 2y'' + y = 0.$$

Solution:

The characteristic equation for this ODE is $r^4 + 2r^2 + 1 = 0$ which factors to $(r^2 + 1)^2 = 0$ giving solutions of $r = \pm i$ each of multiplicity 2. Hence the general solution is

$$y(t) = c_1\, \cos(t) + c_2\, \sin(t) + c_3\, t\, \cos(t) + c_4\, t\, \sin 9t).$$