MATH 214 – EXAM 2 VERSION 1 – SOLUTIONS

1. (10 pts.) Find the solution of the initial value problem

$$y'' - 2y' + y = t\,e^{2t}, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution:

First we need to find the general solution to the homogeneous system. The characteristic equation for this ODE is $r^2 - 2r + 1 = 0$ which factors to $(r-1)^2 = 0$ giving a solution of $r = 1$ of multiplicity 2. Hence the general solution is

$$y(t) = c_1\,e^t + c_2\,t\,e^t.$$

We will use the method of undetermined coefficients to find a particular solution, of the form $Y(t) = A\,t\,e^{2t} + B\,e^{2t}$. This gives $Y'(t) = 2A\,t\,e^{2t} + A\,e^{2t} + 2B\,e^{2t}$ and $Y''(t) = 4A\,t\,e^{2t} + 4A\,e^{2t} + 4B\,e^{2t}$. Plugging into the original equation gives

$$4A\,t\,e^{2t} + 4A\,e^{2t} + 4B\,e^{2t} - 4A\,t\,e^{2t} - 2A\,e^{2t} - 4B\,e^{2t} + A\,t\,e^{2t} + B\,e^{2t} = t\,e^{2t}.$$

Gathering like terms, this leads to

$$A\,t\,e^{2t} + (2A + B)\,e^{2t} = t\,e^{2t}$$

Which solves to $A = 1$, $B = -2$. Hence the general solution is

$$y(t) = c_1\,e^t + c_2\,t\,e^t + t\,e^{2t} - 2\,e^{2t}.$$

Plugging in the initial conditions leads to the system

$$\begin{aligned} 1 &= c_1 - 2 \\ 0 &= c_1 + c_2 + 1 - 4 \end{aligned}$$

yielding a solution of $c_1 = 3$ and $c_2 = 0$. Hence the final solution is

$$y(t) = 3\,e^t + t\,e^{2t} - 2\,e^{2t}.$$

2. (10 pts.) Use the method of variation of parameters to find a particular solution to the differential equation

$$y'' - y' - 2y = 2\,e^{-t}.$$

Solution:

First we need to find the general solution to the homogeneous system. The characteristic equation for this ODE is $r^2 - r - 2 = 0$ which factors to $(r+1)(r-2) = 0$ giving solutions of $r = -1$ and $r = 2$. Hence the general solution is

$$y(t) = c_1\, e^{-t} + c_2\, e^{2t}.$$

We will use the method of variation of parameters to find a particular solution, of the form $Y(t) = u_1(t)\, e^{-t} + u_2(t)\, e^{2t}$. The variation of parameters formula gives the system

$$\begin{aligned} u_1'(t)\, e^{-t} + u_2'(t)\, e^{2t} &= 0 \\ -u_1'(t)\, e^{-t} + 2u_2'(t)\, e^{2t} &= 2\, e^{-t} \end{aligned}$$

Adding the equations gives $3u_2'(t)\, e^{2t} = 2\, e^{-t}$ which solve to $u_2'(t) = (2/3)e^{-3t}$ or $u_2(t) = -(2/9)\, e^{-3t}$. Plugging into the first equation gives $u_1'(t) = -2/3$ or $u_1 = -(2/3)t$. Hence a particular solution is given by

$$Y(t) = -\frac{2}{3}\, t\, e^{-t} - \frac{2}{9}\, e^{-t}.$$

The second term in that sum could be omitted since it is a solution to the homogeneous equation.

3. (10 pts.) Find the general solution to the differential equation

$$y''' - y'' = 6t.$$

Solution:

First we need to find the general solution to the homogeneous system. The characteristic equation for this ODE is $r^3 - r^2 = 0$ which factors to $r^2(r-1) = 0$ giving solutions of $r = 0$ of multiplicity 2 and $r = 1$. Hence the general solution is

$$y(t) = c_1 + c_2\, t + c_3\, e^t + Y(t)$$

where $Y(t)$ is a particular solution. We will find $Y(t)$ by the method of undetermined coefficients. If $Y(t) = At^3 + Bt^2$ then $Y'(t) = 3At^2 + 2Bt$, $Y''(t) = 6At + 2B$, and $Y'''(t) = 6A$. Plugging into the original equation gives

$$6A - 6At + 2B = 6t$$

which gives $A = -1$ and $B = 3$. Hence the general solution to the equation is

$$y(t) = c_1 + c_2\, t + c_3\, e^t - t^3 + 3t^2.$$

4. (10 pts.) Find the general solution to the differential equation

$$y^{(4)} - 5y'' + 4y = 0.$$

Solution:

The characteristic equation for this ODE is $r^4 - 5r^2 + 4 = 0$ which factors to $(r^2 - 1)(r^2 - 4) = 0$ and again to $(r-1)(r+1)(r-2)(r+2) = 0$ giving solutions of $r = 1,\ -1,\ 2,\ -2$. Hence the general solution is

$$y(t) = c_1\, e^t + c_2\, e^{-t} + c_3\, e^{2t} + c_4\, e^{-2t}.$$