MATH 214 – EXAM 1 VERSION 2 – SOLUTIONS

1. (5 pts. each) Consider the initial value problem $y' = y^2(y-2)$, $y(0) = y_0$.

(a) Find all equilibrium solutions to this equation. Do not attempt to solve the IVP.

(b) Describe the long term behavior (that is, the behavior as $t \to \infty$) of the solutions to the IVP for various values of y_0. Give as complete a description as possible. You may sketch a direction field to help you solve this problem but it is not necessary.

Solution:

(a) Equilibrium solutions are $y = 0$ and $y = 2$.

(b) If $y_0 < 0$ then $y' < 0$ so that as $t \to \infty$, $y(t) \to -\infty$. If $0 < y_0 < 2$ then $y' < 0$ so that as $t \to \infty$, $y(t) \to 0$. If $2 < y_0$ then $y' > 0$ so that as $t \to \infty$, $y(t) \to \infty$.

2. (5 pts.) A bathtub initially contains 30 gallons of clean water. Salt water with a concentration of 20 grams of salt per gallon is poured into the tub at a rate of 4 gallons per minute and the drain is opened to drain the tub at the same rate. Set up and solve an initial value problem giving $Q(t)$, the amount of salt in the tub at time t.

Solution:

The correct equation has the form $Q'(t) =$ rate in $-$ rate out. Since the concentration of the solution coming in is 20 grams per gallon and the rate is 4 gallons per minute, salt is coming in to the tub at a rate of $(20)(4) = 80$ grams per minute. Since the concentration of the solution flowing out is $Q(t)/30$ grams per gallon and the rate is 4 gallons per minute, salt is flowing out of the tub at a rate of $(4)(Q(t)/30) = (2/15)Q(t)$ grams per minute. Hence we are solving the equation

$$Q'(t) = 80 - \frac{2}{15}Q(t).$$

Using the integrating factor $\mu(t) = e^{(2/15)t}$ we have

$$\begin{aligned} Q'(t) + \frac{2}{15}Q(t) &= 80 \\ (e^{(2/15)t}Q)' &= 80\, e^{(2/15)t} \\ e^{(2/15)t}Q(t) &= 600\, e^{(2/15)t} + c \\ Q(t) &= 600 + c\, e^{-(2/15)t} \end{aligned}$$

Since $Q(0) = 0$ this gives $c = -600$ so that the final solution is

$$Q(t) = 600 - 600\, e^{-(2/15)t} = 600(1 - e^{-(2/15)t}).$$

3. (5 pts.) Suppose that the field mouse population, $p(t)$, in a certain field satisfies the differential equation $\dfrac{dp}{dt} = p - 800$, where t is measured in years. If the initial population $p(0) = 600$, solve the initial value problem and find the time T at which the population becomes extinct.

Solution:

Solving the equation using the integrating factor $\mu(t) = e^{-t}$ we have

$$\begin{aligned} \frac{dp}{dt} - p &= -800 \\ (e^{-t}p)' &= -800\, e^{-t} \\ e^{-t}p(t) &= 800\, e^{-t} + c \\ p(t) &= 800 + c\, e^{t} \end{aligned}$$

Since $p(0) = 600$ this gives $c = -200$ so that the final solution is

$$p(t) = 800 - 200\, e^{t}.$$

The population will reach zero when T satisfies $p(T) = 0$, or equivalently when $800 - 200\, e^{T} = 0$ which leads to a solution of $T = \ln(4)$.

4. (5 pts. each) Solve each of the following problems.

(a) $y\,\dfrac{dy}{dt} = t^2$, $y(0) = 2$.

(b) $\dfrac{dy}{dt} - 3y = 6e^{t}$, $y(0) = -2$.

(c) $(3x^2 + 2xy) + (2y + x^2)\, y' = 0$. (Hint: This equation is exact.)

Solution:

(a) This equation is separable and we have

$$\begin{aligned} y\,\frac{dy}{dt} &= t^2 \\ y\,dy &= t^2\,dt \\ \frac{1}{2}y^2 &= \frac{1}{3}t^3 + c. \end{aligned}$$

The initial condition $y(0) = 2$ leads to the equation $(1/2)(2^2) = (1/3)(0) + c$ or equivalently $c = 2$. Hence the final solution is given by the equation

$$\frac{1}{2}y^2 = \frac{1}{3}t^3 + 2.$$

(b) Solving the equation using the integrating factor $\mu(t) = e^{-3t}$ we have

$$\begin{aligned} \frac{dy}{dt} - 3y &= 6e^t \\ (e^{-3t}y)' &= 6e^t e^{-3t} \\ (e^{-3t}y)' &= 6e^{-2t} \\ e^{-3t}y(t) &= -3e^{-2t} + c \\ y(t) &= -3e^t + c\,e^{3t}. \end{aligned}$$

The initial condition $y(0) = -2$ leads to the equation $-2 = -3 + c$ or $c = 1$. Hence the final solution is

$$y(t) = -3e^t + e^{3t}.$$

(c) We are looking for a function $\Psi(x, y)$ such that $\Psi_x = 3x^2 + 2xy$ and $\Psi_y = 2y + x^2$. This leads to

$$\begin{aligned} \Psi_x &= 3x^2 + 2xy \\ \Psi &= x^3 + x^2y + g(y) \\ \Psi_y &= x^2 + g'(y). \end{aligned}$$

This leads to

$$\begin{aligned} x^2 + g'(y) &= 2y + x^2 \\ g'(y) &= 2y \\ g(y) &= y^2. \end{aligned}$$

Therefore $\Psi = x^3 + x^2y + y^2$ and the solution to the ODE is given by

$$x^3 + x^2y + y^2 = c.$$

5. (5 pts.) Find an interval of t on which the solution to the initial value problem $(4 - t)y' + 2ty = 3t^2$, $y(-3) = 1$ is certain to exist. Do not solve the IVP!

Solution:

Putting the equation into standard form gives

$$y' + \frac{2t}{4-t}y = \frac{3t^2}{4-t}.$$

The coefficient functions $p(t) = \dfrac{2t}{4-t}$ and $q(t) = \dfrac{3t^2}{4-t}$ are continuous on the intervals $(-\infty, 4)$ and $(4, \infty)$. Since the initial condition $y(-3) = 1$, has t_0-$3 \in (-\infty, 4)$, we can guarantee that the solution exists on the interval $(-\infty, 4)$.