MATH 214 – EXAM 1 VERSION 1 – SOLUTIONS

1. (5 pts. each) Consider the initial value problem $y' = y(y-2)^2$, $y(0) = y_0$.

(a) Find all equilibrium solutions to this equation. Do not attempt to solve the IVP.

(b) Describe the long term behavior (that is, the behavior as $t \to \infty$) of the solutions to the IVP for various values of y_0. Give as complete a description as possible. You may sketch a direction field to help you solve this problem but it is not necessary.

Solution:

(a) Equilibrium solutions are $y = 0$ and $y = 2$.

(b) If $y_0 < 0$ then $y' < 0$ so that as $t \to \infty$, $y(t) \to -\infty$. If $0 < y_0 < 2$ then $y' > 0$ so that as $t \to \infty$, $y(t) \to 2$. If $2 < y_0$ then $y' > 0$ so that as $t \to \infty$, $y(t) \to \infty$.

2. (5 pts.) 20 grams of an undesirable chemical has been introduced into a bathtub containing 30 gallons of water. You turn on the spigot and allow uncontaminated water to come in at a rate of 4 gallons per minute and open the drain to let the water out at the same rate. Set up and solve an initial value problem giving $Q(t)$, the amount of chemical in the tub at time t.

Solution:

The correct equation has the form $Q'(t) = \text{rate in} - \text{rate out}$. Since the concentration of the chemical in the water coming into the tub from the spigot is zero, the rate of chemical coming in to the tub is zero. Since the concentration of the chemical in the bath water flowing out is $Q(t)/30$ grams per gallon and the rate is 4 gallons per minute, the chemical is flowing out of the tub at a rate of $(4)(Q(t)/30) = (2/15)Q(t)$ grams per minute. Hence we are solving the equation

$$Q'(t) = -\frac{2}{15}Q(t).$$

The solution is given by

$$Q(t) = c\,e^{-(2/15)t}.$$

Since $Q(0) = 20$, this gives $c = 20$ so that the final solution is

$$Q(t) = 20\,e^{-(2/15)t}.$$

3. (5 pts.) Suppose that a person takes out a loan at an *annual* interest rate is 6%, and on which he pays \$100 *per month*. Suppose that the interest rate is compounded continuously and that the payments are made continuously. Set up and solve an initial value problem giving the remaining balance on the loan $A(t)$ if the initial amount is $A(0) = A_0$.

Solution:

There are two solutions to this problem depending on whether we measure t in months or years. If t is measured in months then the interest rate r is $.06/12 = .005$ and the rate at which the loan is being paid off is 100 dollars per month. Hence the equation we are solving is

$$A'(t) = .005\,A(t) - 100.$$

Using the integrating factor $\mu(t) = e^{-.005t}$ we have

$$\begin{aligned}
A'(t) &= .005\,A(t) - 100 \\
A'(t) - .005\,A(t) &= -100 \\
(e^{-.005t}A)' &= -100\,e^{-.005t} \\
e^{-.005t}A(t) &= \frac{100}{.005}\,e^{-.005t} + c \\
A(t) &= 20000 + c\,e^{.005t}
\end{aligned}$$

The initial condition $A(0) = A_0$ gives $A_0 = 20000 + c$ so that $c = A_0 - 20000$. Hence the final solution in this case is

$$A(t) = 20000 + (A_0 - 20000)\,e^{.005t} = A_0\,e^{.005t} + 20000(1 - e^{.005t}).$$

If t is measured in years then the interest rate r is .06 and the rate at which the loan is being paid off is $(12)(100) = 1200$ dollars per year. Hence the equation we are solving is

$$A'(t) = .06\,A(t) - 1200.$$

Using the integrating factor $\mu(t) = e^{-.06t}$ we have

$$\begin{aligned}
A'(t) &= .06\,A(t) - 1200 \\
A'(t) - .06\,A(t) &= -1200 \\
(e^{-.06t}A)' &= -1200\,e^{-.06t} \\
e^{-.06t}A(t) &= \frac{1200}{.06}\,e^{-.06t} + c \\
A(t) &= 20000 + c\,e^{.06t}
\end{aligned}$$

The initial condition $A(0) = A_0$ gives $A_0 = 20000 + c$ so that $c = A_0 - 20000$. Hence the final solution in this case is

$$A(t) = 20000 + (A_0 - 20000)\,e^{.06t} = A_0\,e^{.06t} + 20000(1 - e^{.06t}).$$

4. (5 pts. each) Solve each of the following problems.

(a) $\dfrac{dy}{dt} - 2y = 4$, $y(0) = 2$.

(b) $\dfrac{dy}{dt} - 2y = 4e^t$, $y(0) = -2$.

(c) $(3x^2 - 2xy) + (y - x^2)\, y' = 0$. (Hint: This equation is exact.)

Solution:

(a) Using the integrating factor $\mu(t) = e^{-2t}$ we have

$$\begin{aligned} \frac{dy}{dt} - 2y &= 4 \\ (e^{-2t}\, y)' &= 4e^{-2t} \\ e^{-2t}\, y &= -2\, e^{-2t} + c \\ y(t) &= -2 + c\, e^{2t}. \end{aligned}$$

The initial condition $y(0) = 2$ leads to the equation $2 = -2 + c$ or $c = 4$. Hence the final solution is given by the equation

$$y(t) = -2 + 4\, e^{2t}.$$

(b) Solving the equation using the integrating factor $\mu(t) = e^{-2t}$ we have

$$\begin{aligned} \frac{dy}{dt} - 2y &= 4e^t \\ (e^{-2t}y)' &= 4e^t e^{-2t} \\ (e^{-2t}y)' &= 4e^{-t} \\ e^{-2t}y(t) &= -4e^{-t} + c \\ y(t) &= -4e^t + c\, e^{2t}. \end{aligned}$$

The initial condition $y(0) = -2$ leads to the equation $-2 = -4 + c$ or $c = 2$. Hence the final solution is

$$y(t) = -4e^t + 2\, e^{2t}.$$

(c) We are looking for a function $\Psi(x,y)$ such that $\Psi_x = 3x^2 - 2xy$ and $\Psi_y = y - x^2$. This leads to

$$\begin{aligned}
\Psi_x &= 3x^2 - 2xy \\
\Psi &= x^3 - x^2 y + g(y) \\
\Psi_y &= -x^2 + g'(y).
\end{aligned}$$

This leads to

$$\begin{aligned}
-x^2 + g'(y) &= y - x^2 \\
g'(y) &= y \\
g(y) &= \frac{1}{2}y^2.
\end{aligned}$$

Therefore $\Psi = x^3 - x^2y + (1/2)y^2$ and the solution to the ODE is given by

$$x^3 - x^2y + (1/2)y^2 = c.$$

5. (5 pts.) Find an interval of t on which the solution to the initial value problem $(4-t)y' + 2ty = 3t^2$, $y(-3) = 1$ is certain to exist. Do not solve the IVP!

Solution:

Putting the equation into standard form gives

$$y' + \frac{2t}{4-t}y = \frac{3t^2}{4-t}.$$

The coefficient functions $p(t) = \dfrac{2t}{4-t}$ and $q(t) = \dfrac{3t^2}{4-t}$ are continuous on the intervals $(-\infty, 4)$ and $(4, \infty)$. Since the initial condition $y(-3) = 1$, has t_0-$3 \in (-\infty, 4)$, we can guarantee that the solution exists on the interval $(-\infty, 4)$.