1.1 Basic Models; Direction Fields.

A. Start by considering $\frac{dy}{dt} = f(t,y)$; $y = y(t)$

Solution to this equation is a function of t.

Any solution $y = y(t)$ satisfies $y'(t) = f(t, y(t))$ for all t.

B. Consider the example $\frac{dy}{dt} = y$

1. Solution has property that its derivative is itself. $y(t) = e^t$

2. Is this the only solution?

$$\boxed{y(t) = e^t + 2 \quad \frac{dy}{dt} = e^t}$$

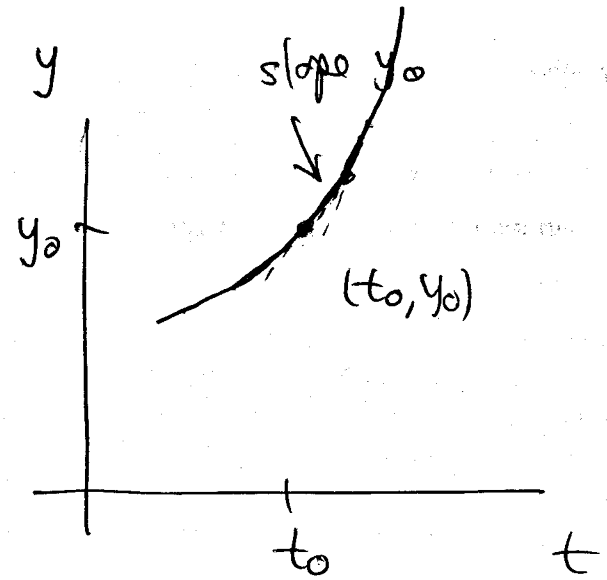
$$y(t) = 2e^t \quad \frac{dy}{dt} = 2e^t$$

$$y(t) = 0$$

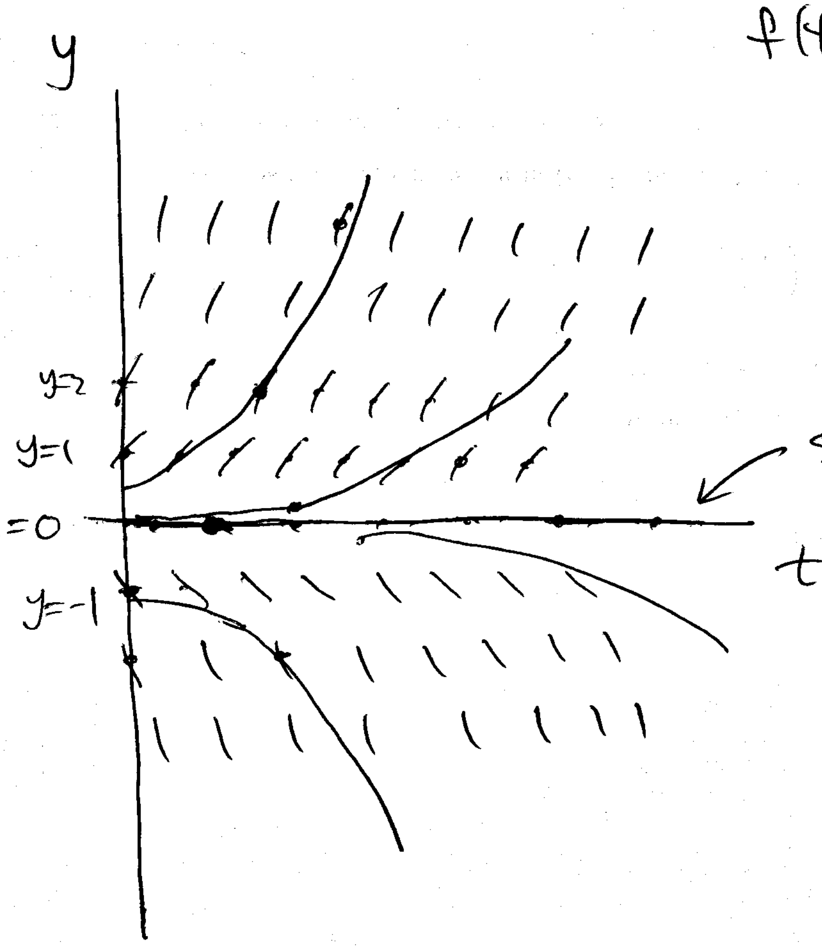
3. All solutions have the form $y = ce^t$ where c = constant.

4. Consider another approach.

$\frac{dy}{dt} = y$ means that the slope of $y(t)$ is same as the value of $y(t)$



5. Direction Field. In our example $f(t,y) = y$.



At each (t,y) draw a line segment with slope $f(t,y) = y$.

solution $y=0$ is an equilibrium solution.

(this equilibrium solution is repelling)

C. Example: Falling object.

t = time in seconds

v = velocity in m/sec.

m = mass in kg

γ = drag coefficient in kg/sec
(depends on mass + shape of object)

$g = 9.8$ m/sec^2 acceleration due to gravity

$$m \frac{dv}{dt} = mg - \gamma v$$

mass × acceleration = force

force due to gravity

force of air resistance

$\frac{dv}{dt} > 0$ means object is falling

$$\frac{dv}{dt} = g - \frac{\gamma}{m} v$$

e.g. $g = 9.8 \quad \gamma = 2 \quad m = 10$

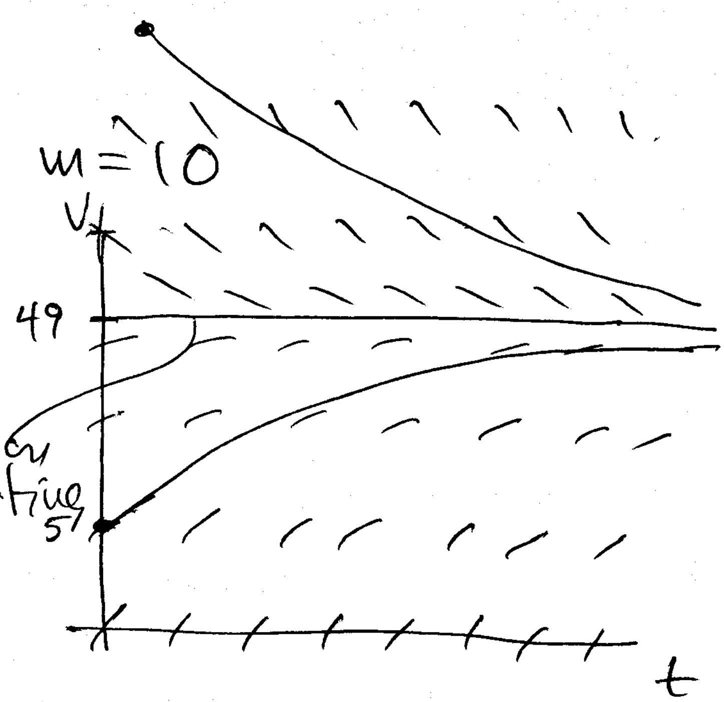
$$\frac{dv}{dt} = 9.8 - .2v$$

Equilibrium soln:

$9.8 - .2v = 0$

$v = \frac{9.8}{.2} = 49 \frac{m}{sec}$

solution is attractive



D. Example: Of Mice and Owls.

$t =$ time in months

$p =$ mouse population (in mice)

$r =$ rate constant or growth rate

$\frac{dp}{dt} = rp$ (growth model in absence of predation)

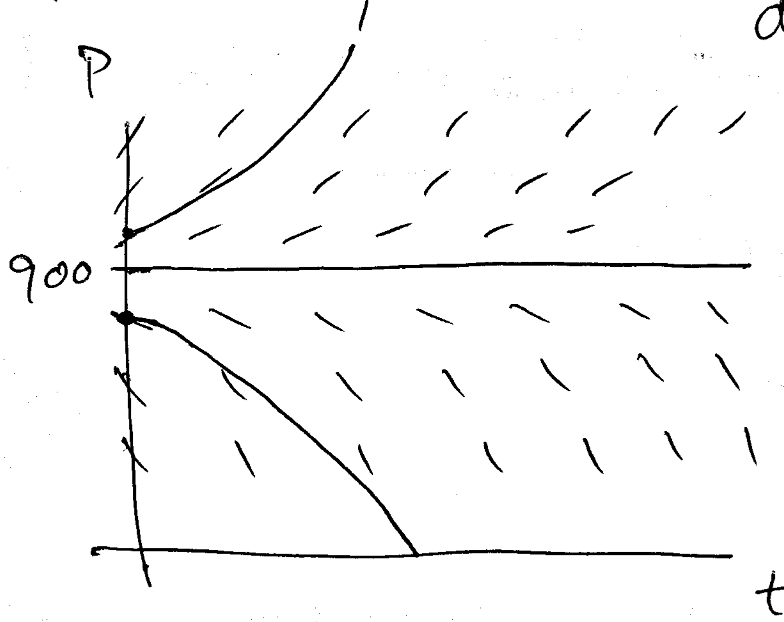
Assume fixed rate of predation by owls. That is owls eat k mice/month.

$\frac{dp}{dt} = rp - k$

If $r = .5$, $k = 450$ $\quad \frac{dp}{dt} = .5p - 450$

$.5p - 450 = 0$

$p = 900$



#20) 2 equilibrium solutions

$y = 0$, $y = 3$

(e) $y' = y(y-3)$ (h) $y' = y(3-y)$

$y = 0$: attracting

$y = 3$: repelling.

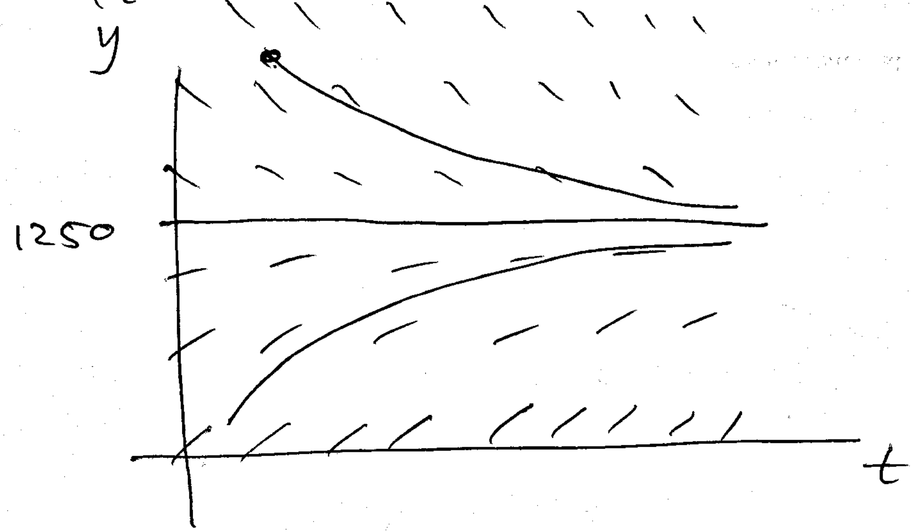
(e) $y' = y(y-3)$.

#24) t = time in hours

y = amount present in bloodstream in mg.

$$\frac{dy}{dt} = \binom{\text{rate}}{\text{in}} - \binom{\text{rate}}{\text{out}}$$

$$\frac{dy}{dt} = 500 - .4y$$



$500 - .4y = 0$

$y = \frac{500}{.4} = 1250\text{mg}$

As $t \to +\infty$

$y(t) \to \underline{\underline{1250}}$.

1.2 Solutions of some ODEs.

To solve $\frac{dy}{dt} = f(t,y)$ we can sometimes find an explicit solution.

A. $\frac{dp}{dt} = .5p - 450 = \frac{p-900}{2}$

$$\frac{dp/dt}{p-900} = \frac{1}{2} \longrightarrow \frac{dp}{p-900} = \frac{1}{2}\,dt$$

(separation of variables.)

$$\int \frac{dp}{p-900} = \int \frac{1}{2}\,dt$$

$$\ln|p-900| = \frac{1}{2}t + C$$

$$|p-900| = e^{\frac{1}{2}t + C} = e^{C} e^{\frac{1}{2}t}$$

$$p - 900 = \boxed{\pm e^{C}}\, e^{\frac{1}{2}t}$$

call it c

$$\boxed{p = 900 + ce^{\frac{1}{2}t}}$$

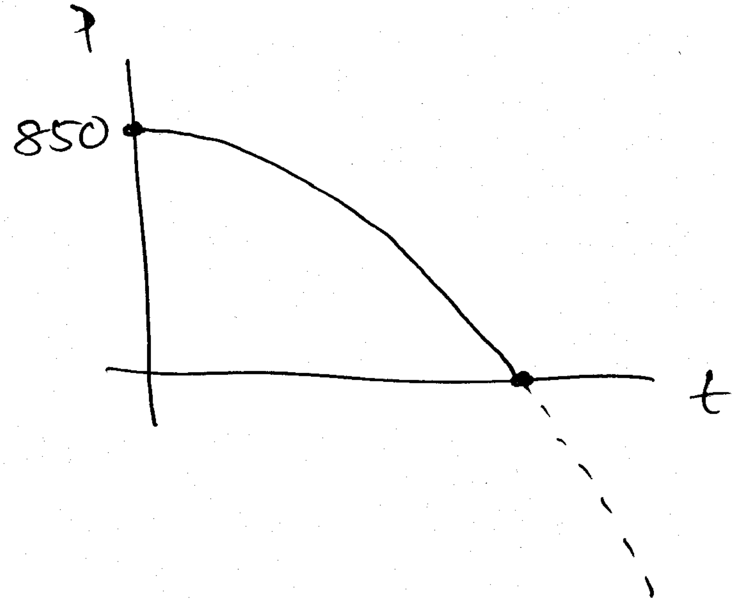
what is c? $p(0) = 900 + ce^{0} = 900 + c$

initial population

$c = p(0) - 900$

e.g. $p(0) = 850 \qquad c = -50$

$$p(t) = 900 - 50e^{\frac{1}{2}t}$$



$p = 0$ at some point.

When?

$$0 = 900 - 50e^{\frac{1}{2}t}$$

$$50e^{\frac{1}{2}t} = 900$$

$$e^{\frac{1}{2}t} = 18$$

$$t = 2\ln 18$$

$$= 5.8 \text{ months.}$$

B. $\dfrac{dv}{dt} = 9.8 - .2v = \dfrac{49 - v}{5}$

$$\frac{d}{dt}\ln|t| = \frac{1}{t}$$

$$\int \frac{dv}{49 - v} = \int \frac{1}{5}\,dt$$

$$-\ln|49 - v| = \frac{1}{5}t + c$$

$$\ln|49 - v| = -\frac{1}{5}t - c$$

$$|49 - v| = e^{-\frac{1}{5}t}e^{-c}$$

$$49 - v = ce^{-\frac{1}{5}t}$$

$$v = 49 - \underline{\underline{c}}\,e^{-\frac{1}{5}t}$$