Solving 2nd order, linear, homogeneous equations

In general: $y'' + p(t)y' + q(t)y = 0 \rightarrow L[y] = 0$.

1. Because equation is linear and homogeneous we know: If y_1, y_2 solve $L[y_1] = L[y_2] = 0$, then $L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2] = 0$.

2. Solution method:
   a. Find 2 solutions $y_1(t)$, $y_2(t)$.
   b. Check that $W(y_1, y_2)(t) \neq 0$ for some t in the interval where solutions exist. (This means y_1 and y_2 are linearly independent)
   c. Any solution has the form $y(t) = c_1 y_1(t) + c_2 y_2(t)$
   d. In this case we can always find c_1 and c_2 satisfying initial conditions $y(t_0) = y_0$, $y'(t_0) = y_0'$.

3. Currently looking at step 2a. for constant coefficients, i.e. $ay'' + by' + cy = 0$.

Complex roots.

#8 ~~y''~~ $y'' - 2y' + 6y = 0$.

$r^2 - 2r + 6 = 0$

$r = \frac{2 \pm \sqrt{4-24}}{2} = \frac{2 \pm \sqrt{-20}}{2} = \frac{2 \pm 2\sqrt{-5}}{2} = 1 \pm i\sqrt{5}$

Idea: $y_1 = e^{(1+\sqrt{5}i)t}$ $\qquad y_2 = e^{(1-\sqrt{5}i)t}$

$= e^t e^{\sqrt{5}ti}$ $\qquad = e^t e^{-\sqrt{5}ti}$

Know: $\cos(-x) = \cos(x)$, $\sin(-x) = -\sin(x)$

$\boxed{e^{ix} = \cos(x) + i\sin(x)}$ $\qquad = e^t(\cos\sqrt{5}t - i\sin\sqrt{5}t)$

$= e^t(\cos\sqrt{5}t + i\sin\sqrt{5}t)$

Observe: $\frac{1}{2}(y_1 + y_2) = \cancel{2}e^t\cos(\sqrt{5}t)$ ← solns. in fact

$\frac{1}{2i}(y_1 - y_2) = \cancel{2i}e^t\sin(\sqrt{5}t)$ ← $W(y_1, y_2) \neq 0$.

~~Fund~~ General solution $y = c_1 e^t\cos(\sqrt{5}t) + c_2 e^t\sin(\sqrt{5}t)$.

#8) $y'' + 4y' + 5y = 0$

$r^2 + 4r + 5 = 0$

$$r = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i$$

$\underline{y_1(t) = e^{-2t}\cos(t)}$, $\underline{y_2 = e^{-2t}\sin(t)}$.

$$e^{(-2+i)t} \qquad e^{(-2-i)t}$$

$$\| \qquad \|$$

$$e^{-2t}e^{it} \qquad e^{-2t}e^{-it}$$

$$\| \qquad \|$$

$$e^{-2t}(\cos t + i\sin t) \qquad e^{-2t}(\cos t - i\sin t)$$

Initial conditions: $y(0) = 1 \quad y'(0) = 0$

$y(t) = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t)$

$y'(t) = -c_1 e^{-2t}\sin t - 2c_1 e^{-2t}\cos t$
$\qquad + c_2 e^{-2t}\cos t - 2c_2 e^{-2t}\sin t$

$$1 = c_1 \qquad\qquad c_1 = 1$$
$$0 = -2c_1 + c_2 \longrightarrow c_2 = 2$$

$$y(t) = e^{-2t}\cos t + 2e^{-2t}\sin t.$$

3.4 Repeated Roots

e.g. $y'' + 4y' + 4y = 0.$

$r^2 + 4r + 4 = 0.$

$(r+2)(r+2) = 0$

$(r+2)^2 = 0$

$r = -2 \longrightarrow \underline{y_1(t) = e^{-2t}}$ is a solution

How do we find y_2? Take a guess that y_2 has the form $y_2(t) = v(t)e^{-2t}$, try to find $v(t)$.

$$y_2'(t) = -2v(t)e^{-2t} + v'(t)e^{-2t}$$

$$y_2''(t) = 4v(t)e^{-2t} - 2v'(t)e^{-2t} - 2v'(t)e^{-2t} + v''(t)e^{-2t}$$
$$= 4v(t)e^{-2t} - 4v'(t)e^{-2t} + v''(t)e^{-2t}$$

$$y_2'' + 4y_2' + 4y_2 = 0 \text{ gives.}$$

$$4ve^{-2t} - 4v'e^{-2t} + v''e^{-2t} + 4(-2ve^{-2t} + v'e^{-2t}) + 4ve^{-2t} = 0$$

$$\cancel{4v} - \cancel{4v'} + v'' - \cancel{8v} + \cancel{4v'} + \cancel{4v} = 0$$

$$v'' = 0 \longrightarrow v = c_1 t + c_2$$

So $y_2(t) = (c_1 t + c_2)e^{-2t} = c_1 t e^{-2t} + \underbrace{c_2 e^{-2t}}_{\text{multiple of } y_1}$

only need $\underline{a}$ solution so take $c_1 = 1$

So take $\underline{y_2(t) = te^{-2t}}$

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

$$= e^{-2t}(\cancel{-2te^{-2t}} + e^{-2t}) - \cancel{te^{-2t}(-2e^{-2t})}$$

$$= e^{-4t} \neq 0.$$

This will always work!

Solve $ay'' + by' + cy = 0$

$ar^2 + br + c = a(r - r_1)^2$

$y_1 = e^{r_1 t}$ is one solution

$y_2 = te^{r_1 t}$ is another solution ~~in~~

and $W(y_1, y_2) \neq 0$.

Letting $y_2 = ve^{r_1 t}$ always leads to $v'' = 0$ so $v(t) = t$ always works

e.g. #12 $y'' - 6y' + 9y = 0$ $y(0) = 0$ $y'(0) = 2$.

$r^2 - 6r + 9 = 0$

$(r - 3)^2 = 0$

$r = 3$ $y_1(t) = e^{3t}$ $y_2(t) = te^{3t}$

$y(t) = c_1 e^{3t} + c_2 te^{3t}$

Find c_1, c_2. $y'(t) = 3c_1 e^{3t} + 3c_2 te^{3t} + c_2 e^{3t}$

$0 = c_1$ $\rightarrow$ $c_1 = 0$

$2 = 3c_1 + c_2$ $\quad$ $c_2 = 2$

$y(t) = 2te^{3t}$ //

This technique gives us a way to solve the general equation $L[y] = y'' + p(t)y' + q(t)y = 0$.
If we already have y_1 solving $L[y_1] = 0$ it is possible to find y_2.

<u>Reduction of order</u>

Suppose $y_1'' + p(t)y_1' + q(t)y_1 = 0$

Look for y_2 of the form $y_2 = v y_1$

$y_2' = v y_1' + v' y_1$; $y_2'' = v y_1'' + 2v'y_1' + v''y_1$

$v y_1'' + 2v'y_1' + v''y_1 + p v y_1' + p v' y_1 + q v y_1 = 0$

$v\underbrace{(y_1'' + p y_1' + q y_1)}_{0} + v'(2y_1' + p y_1) + v'' y_1 = 0$

$$\boxed{\text{Solve: } v'' y_1 + (2y_1' + p y_1) v' = 0}$$

This is a 1st order equation in $u = v'$

If $u = v'$ then $u' = v''$ so equ is

$$u' y_1 + (2y_1' + p y_1) u = 0$$

eg 3 $2t^2y''+3ty'-y=0$

Given $y_1(t)=t^{-1}$ is a solution

Look for $y_2 = vy_1 = v(t)t^{-1}$

$y_2' = -v(t)t^{-2}+v'(t)t^{-1}$

$y_2'' = 2v(t)t^{-3}-2v'(t)t^{-2}+v''(t)t^{-1}$

$2t^2y_2'' = 4t^2v(t)t^{-3}-4t^2v'(t)t^{-2}+2t^2v''(t)t^{-1}$

$= 4t^2v(t)t^{-3}-4v'(t)+2tv''(t)$

$= 4t^{-1}v(t)-4v'(t)+2tv''(t)$

$3ty_2' = -3tv(t)t^{-2}+3tv't^{-1}$

$= 3t^{-1}v(t)+3v'(t)$

$\cancel{4t^{-1}v(t)}-4v'(t)+2tv''(t)-\cancel{3t^{-1}v(t)}+3v'(t)-\cancel{v(t)t^{-1}}=0$

$2tv''-v'=0 \qquad u=v'$

$2tu'-u=0$

$2t\frac{du}{dt}=u$

$\frac{du}{u}=\frac{1}{2t}dt$

$\ln|u|=\frac{1}{2}\ln|t|+c$

$|u|=c|t|^{1/2}$

$u=ct^{1/2}$ ← since $t>0$

→ $v=\int ct^{1/2}dt=\frac{2}{3}ct^{3/2}+k$

only need a solution so take $k=0$, $c=\frac{3}{2}$ so $v=t^{3/2}$

So $y_2 = t^{3/2} \cdot y_1 = t^{3/2} \cdot t^{-1} = t^{1/2}$.

Check $W(y_1, y_2)$.

$$W(y_1,y_2) = y_1 y_2' - y_2 y_1'$$

$$= t^{-1} \tfrac{1}{2} t^{-1/2} - t^{1/2} \cdot (-t^{-2})$$

$$= \tfrac{1}{2} t^{-3/2} + t^{-3/2} = \tfrac{3}{2} t^{-3/2} \neq 0 \text{ for } \underline{t>0}.$$

General solution $y = c_1 t^{-1} + c_2 t^{1/2}$, $\underline{t>0}$.

Another approach:

rewrite eqn as $y'' + \frac{3t}{2t^2} y' - \frac{1}{2t^2} y = 0$

$$\underbrace{y'' + \frac{3}{2t}}_{} y' - \frac{1}{2t^2} y = 0$$

$\frac{3}{2t}$ ↑ $p(t)$ $\qquad$ $\frac{1}{2t^2}$ ↑ $q(t)$