Quiz 13. Sections 6.4, 6.5

Review of Matrices.

Adding / Subtracting

Multiplying

Inverse Matrices.

Transposes.

#14, 7.2) $\begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{pmatrix}$ singular – inverse does not exist (analogous to number 0.)

$$AA^{-1} = I$$

$$\begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{pmatrix}\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Equivalent to solving 3 linear systems:

$$\begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{pmatrix}\begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad \begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{pmatrix}\begin{pmatrix} c_{12} \\ c_{22} \\ c_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix};$$

$$\begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{pmatrix}\begin{pmatrix} c_{13} \\ c_{23} \\ c_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Matrix functions.

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

$$\frac{d\vec{x}}{dt} = \vec{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} \qquad A'(t) = \begin{pmatrix} a_{11}'(t) & \cdots & a_{1n}'(t) \\ \vdots & & \vdots \\ a_{n1}'(t) & \cdots & a_{nn}'(t) \end{pmatrix}$$

Rules of differentiation hold, e.g. the product rule.

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#22)

$$\vec{x}'(t) = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x}(t) \qquad \underline{\vec{x}(t) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t}}$$

$$\vec{x}(t) = \begin{pmatrix} 4e^{2t} \\ 2e^{2t} \end{pmatrix} \qquad \vec{x}'(t) = \begin{pmatrix} 8e^{2t} \\ 4e^{2t} \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}$$

More directly:

$$\vec{x}'(t) = \frac{d}{dt}\begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \frac{d}{dt} e^{2t} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \cdot 2e^{2t}$$

$$= \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}.$$

$$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x}(t) = \underbrace{\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix}} e^{2t}$$

$$= \begin{pmatrix} 12-4 \\ 8-4 \end{pmatrix} e^{2t} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}$$

Solve the system:

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \rightarrow \begin{array}{l} x_1' = 3x_1 - 2x_2 \\ x_2' = 2x_1 - 2x_2 \end{array} \rightarrow x_1'' = 3x_1' - 2x_2'$$

$$\rightarrow x_1'' = 3x_1' - 4x_1 + 4x_2$$

$$\cancel{2x_2 = 2x_1 - x_2'}$$
$$\cancel{4x_2 = 4x_1 - 2x_2'}$$

$$x_1'' = 3x_1' - 4x_1 + 6x_1 - 2x_1' \qquad 2x_2 = 3x_1 - x_1'$$

$$= x_1' + 2x_1 \qquad 4x_2 = 6x_1 - 2x_1'$$

$$x_1'' - x_1' - 2x_1 = 0$$

$$r^2 - r - 2 = 0$$

$$(r+1)(r-2) = 0$$

$$r = -1 \quad r = 2$$

$$\underline{x_1 = c_1 e^{-t} + c_2 e^{2t}}$$

$$x_2 = \frac{3}{2} x_1 - \frac{1}{2} x_1'$$

$$= \frac{3}{2}(c_1 e^{-t} + c_2 e^{2t})$$

$$- \frac{1}{2}(-c_1 e^{-t} + 2c_2 e^{2t})$$

$$= \underline{2c_1 e^{-t} + \frac{1}{2} c_2 e^{2t}}$$

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} + c_2 e^{2t} \\ 2c_1 e^{-t} + \frac{1}{2} c_2 e^{2t} \end{pmatrix}$$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} e^{2t}$$

Our solution $\begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t}$ corresponds to

$c_1 = 0 \quad c_2 = 4$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ an eigenvector? $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

$= (-1) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

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## 7.3 Systems of Equations; Linear Independence, Eigenvalues/Eigenvectors.

Systems of Equations:

$$A\vec{x} = \vec{b} \qquad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}; \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}; \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Basic theory: 2 possibilities.

1. $A\vec{x} = \vec{b}$ has a unique solution.
   a. A is non-singular
   b. $A^{-1}$ exists $\longrightarrow \vec{x} = A^{-1}\vec{b}$
   c. $\det(A) \neq 0$.

2. $A\vec{x}=\vec{b}$ does not have a unique solution (this means $A$ is singular, $\det A = 0$).

a. $A\vec{x} = \vec{b}$ has no solutions.

b. $A\vec{x} = \vec{b}$ has infinitely many solutions. (That is, if $A\vec{x} = \vec{b}$ has one solution, it has infinitely many.)

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$\xi\,\xi\,\xi$

3. c. In this case, solutions to $A\vec{x} = \vec{b}$ have the form $\vec{x} = \vec{x}^{(0)} + \vec{\xi}$

$\xi \leftarrow$ xi

where $\vec{x}^{(0)}$ is a particular solution, i.e. $A\vec{x}^{(0)} = \vec{b}$ and $\vec{\xi}$ satisfies $\underline{A\vec{\xi} = \vec{0}}$.

3. $A\vec{x} = \vec{0}$ (homogeneous system) always has the solution $\vec{x} = \vec{0}$. If in addition $A$ is singular, $A\vec{x} = \vec{0}$ has $\infty$-many solutions.

__e.g. 3__

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ -x_1 + x_2 - 2x_3 &= 0 \\ 2x_1 - x_2 + 3x_3 &= 0 \end{aligned} \rightarrow \begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2 \left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -1 & 1 & -2 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right) \rightarrow \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix} \rightarrow$$

$$\left(\begin{array}{cccc} 1 & -2 & 3 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \begin{array}{r} x_1 - 2x_2 + 3x_3 = 0 \\ -x_2 + x_3 = 0 \end{array}$$

$x_1 = -t$

$x_2 = t$

$x_3 = t \Leftarrow$ free parameter

$x_1 - 2t + 3t = 0 \to x_1 + t = 0$

$-x_2 + t = 0 \to x_2 = t \quad x_1 = -t$

All solutions have the form $\vec{x} = \begin{pmatrix} -t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$t \in \mathbb{R}$.

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e.g.
$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ -x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 - 4x_2 + 6x_3 &= 0 \end{aligned}$$

$$\begin{array}{c} -2 \\ \\ \\ \end{array}\left(\begin{array}{cccc} 1 & -2 & 3 & 0 \\ -1 & 2 & -3 & 0 \\ 2 & -4 & 6 & 0 \end{array}\right) \to \left(\begin{array}{cccc} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\to x_1 - 2x_2 + 3x_3 = 0$

$x_1 = 2s - 3t$

$x_2 = s$

$x_3 = t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

Linear Independence.

e.g. 3 $\quad \vec{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \vec{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \quad \vec{x}^{(3)} = \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix}$

Solve: $c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} + c_3 \vec{x}^{(3)} = \vec{0}$

Clearly $c_1 = c_2 = c_3 = 0$ works.
Are there other solutions? If no, then the vectors are <u>linearly independent</u>. If yes, they are <u>linearly dependent</u>.

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$$c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} + c_3 \vec{x}^{(3)} = \vec{0}$$

$$\Updownarrow$$

$$\begin{pmatrix} | & | & | \\ \vec{x}^{(1)} & \vec{x}^{(2)} & \vec{x}^{(3)} \\ | & | & | \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longleftrightarrow A\vec{c} = \vec{0}.$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 2 & 1 & 1 & 0 \\ -1 & 3 & -11 & 0 \end{array}\right) \xrightarrow{\frac{5}{3}} \begin{pmatrix} 1 & 2 & -4 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 5 & -15 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & -4 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Vectors are <u>linearly dependent</u>.

Find a linear relation among vectors.
(i.e. solve the system) $c_1+2c_2-4c_3=0$

$-3c_2+9c_3=0$

$c_1=-2t$
$c_2=3t$
$c_3=t$

$c_1+2c_2-4c_3=c_1+6t-4t=0$
$-3c_2+9t=0$

$\vec{c}=t\begin{pmatrix}-2\\3\\1\end{pmatrix}$ set $t=1$ (say) then

$$\boxed{-2\vec{x}^{(1)}+3\vec{x}^{(2)}+\vec{x}^{(3)}=0.}$$

Also could check determinant.

$$\det\begin{pmatrix}1&2&-4\\2&1&1\\-1&3&-11\end{pmatrix}=(1)\det\begin{pmatrix}1&1\\3&-11\end{pmatrix}-(2)\det\begin{pmatrix}2&1\\-1&-11\end{pmatrix}$$
$$+(-4)\det\begin{pmatrix}2&1\\-1&3\end{pmatrix}$$

$$=(1)(-14)-2(-21)-4(7)$$
$$=-14+42-28=0 \leftarrow \therefore \text{ linearly dependent.}$$

Linearly independent vector (matrix) functions.

e.g. #14) $\vec{x}^{(1)}(t) = \begin{pmatrix} 2\sin t \\ \sin t \end{pmatrix}$ $\vec{x}^{(2)}(t) = \begin{pmatrix} \sin t \\ 2\sin t \end{pmatrix}$

Solve: $c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t) = \vec{0}$

If $c_1 = c_2 = 0$ is only solution valid for all $t$ then linearly independent. If not, linearly dependent.

this means the left side is identically zero, i.e. zero for all $t$.

$$\det\begin{pmatrix} 2\sin t & \sin t \\ \sin t & 2\sin t \end{pmatrix} = 4\sin^2 t - \sin^2 t$$
$$= 3\sin^2 t$$

This will vanish if e.g. $t = 0$ or $\pi$, etc.

Hence $\vec{x}^{(1)}(t)$ $\vec{x}^{(2)}(t)$ are not linearly independent for $-\infty < t < \infty$.