Exam 2 – Nov. 8, Chapters 3, 4

Solving linear equations (constant coeffs) using Laplace transform.

eg: $y'' + y = \sin(2t) \quad y(0)=2 \quad y'(0)=1$

$$\mathcal{L}\{y''+y\} = \mathcal{L}\{\sin(2t)\}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin(2t)\}.$$

$$s^2\mathcal{L}\{y\} - \underline{y(0)}s - \underline{y'(0)} + \mathcal{L}\{y\} = \frac{2}{s^2+4}$$

$$s^2\mathcal{L}\{y\} - 2s - 1 + \mathcal{L}\{y\} = \frac{2}{s^2+4}$$

$$\mathcal{L}\{y\}(s^2+1) = \frac{2}{s^2+4} + 2s+1$$

$$\mathcal{L}\{y\} = \frac{2}{(s^2+1)(s^2+4)} + \frac{2s+1}{s^2+1}$$

$$= \underbrace{\frac{2}{(s^2+1)(s^2+4)}}_{\text{expand in partial fractions}} + \underbrace{2\frac{s}{s^2+1} + \frac{1}{s^2+1}}_{\text{already in the table}}$$

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$= \frac{(As+B)(s^2+4)+(Cs+D)(s^2+1)}{(s^2+1)(s^2+4)}$$

$$2 = (As+B)(s^2+4)+(Cs+D)(s^2+1)$$

$$2 = As^3+4As+Bs^2+4B+Cs^3+Cs+Ds^2+D$$

$$= s^3(A+C)+s^2(B+D)+s(4A+C)+(4B+D)$$

$$\left.\begin{aligned} A+C&=0 \\ 4A+C&=0 \end{aligned}\right\} \rightarrow A=C=0$$

$$(-1)(B+D=0)$$
$$4B+D=2$$

$$3B=2$$

$$B=\frac{2}{3} \quad D=-\frac{2}{3}$$

$$\mathcal{L}\{y\} = \frac{2}{3}\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4} + 2\frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$= \frac{5}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4} + 2\frac{s}{s^2+1}$$

$$y = \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t) + 2\cos(t)$$

homogeneous solution

particular solution

eg #8)

$$F(s) = \frac{8s^2-4s+12}{s(s^2+4)} = 4\,\frac{2s^2-s+3}{s(s^2+4)}$$

$$\frac{2s^2-s+3}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{A(s^2+4)+s(Bs+C)}{s(s^2+4)}$$

$$2s^2-s+3 = A(s^2+4)+s(Bs+C) \qquad s=0$$

$$3 = A\cdot 4 \longrightarrow A = \frac{3}{4}$$

$$\frac{d}{ds}:\ 4s-1 = 2As + Bs + (Bs+C)$$

$$= 2As + 2Bs + C \qquad s=0:$$

$$-1 = C$$

$$\frac{d}{ds}:\ 4 = 2A+2B$$

$$2 = A+B \longrightarrow B = 2-\frac{3}{4} = \frac{5}{4}$$

$$F(s) = 4\left(\frac{3}{4}\cdot\frac{1}{s} + \frac{\frac{5}{4}s-1}{s^2+4}\right) = 3\cdot\frac{1}{s} + \frac{5s-4}{s^2+4}$$

$$= 3\cdot\frac{1}{s} + 5\cdot\frac{s}{s^2+4} - \frac{4}{2}\cdot\frac{2}{s^2+4}$$

$$\mathcal{L}^{-1}(F(s)) = 3 + 5\cos(2t) - 2\sin(2t)$$

eg #22) $y'' - 2y' + 2y = e^{-t}$ $y(0) = 0$ $y'(0) = 1$

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

$$s^2\mathcal{L}\{y\} - s\cancel{y(0)}^{0} - y'(0) - 2\left(s\mathcal{L}\{y\} - \cancel{y(0)}^{0}\right) + 2\mathcal{L}\{y\} = \frac{1}{s+1}$$

$$s^2\mathcal{L}\{y\} - 1 - 2s\mathcal{L}\{y\} + 2\mathcal{L}\{y\} = \frac{1}{s+1}$$

$$\mathcal{L}\{y\}(s^2 - 2s + 2) = \frac{1}{s+1} + 1$$

$$\mathcal{L}\{y\} = \frac{1}{(s+1)(s^2-2s+2)} + \frac{1}{(s^2-2s+2)}$$

$$s^2 - 2s + 2 = (s^2 - 2s + 1) + 1 = (s-1)^2 + 1$$

$$= \frac{1}{(s+1)((s-1)^2+1)} + \frac{1}{(s-1)^2+1}$$ ← in the table

$$\frac{1}{(s+1)((s-1)^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{(s-1)^2+1}$$

$$= \frac{A((s-1)^2+1) + (s+1)(Bs+C)}{(s+1)((s-1)^2+1)}$$

$$1 = A((s-1)^2+1) + (s+1)(Bs+C) \qquad s=-1$$

$$1 = 5A \longrightarrow A = \frac{1}{5} //$$

$$\frac{d}{ds}: \; 0 = A(2(s-1)) + B(s+1) + (Bs+C) \qquad s=1.$$

$$0 = 3B + C \longrightarrow 0 = -\frac{3}{5} + C \rightarrow C = \frac{3}{5} //$$

$$\frac{d^2}{ds}: \; 0 = 2A + 2B$$

$$A + B = 0 \longrightarrow \frac{1}{5} + B = 0 \rightarrow B = -\frac{1}{5} //$$

$$\mathcal{L}\{y\} = \frac{1}{5} \cdot \frac{1}{s+1} + \frac{-\frac{1}{5}s + \frac{3}{5}}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}$$

$$= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{(s-1)}{(s-1)^2+1} + \frac{7}{5}\frac{1}{(s-1)^2+1}$$

$$\frac{-\frac{1}{5}s + \frac{3}{5}}{(s-1)^2+1} = \frac{-\frac{1}{5}s + \frac{1}{5} + \frac{2}{5}}{(s-1)^2+1} = -\frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{2}{5}\frac{1}{(s-1)^2+1}$$

$$y = \frac{1}{5}e^{-t} - \frac{1}{5}e^{t}\cos(t) + \frac{7}{5}e^{t}\sin(t) //$$

6.3 Step Functions

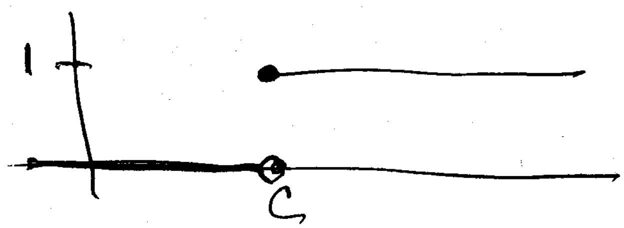
$\mathcal{L}\{f(t)\}$ is defined even if $f(t)$ has jump discontinuities. This means we can solve $ay''+by'+cy=g(t)$ even when $g(t)$ has jumps.

Idea: Why would g have jumps?

1) linear 2nd order equations model oscillatory systems, e.g. mass and spring systems, pendulums, electrical circuits.
2) $ay''+by'+c=0$ describes system with no external forces acting on it.
3) $g(t)$ is called a forcing term and describes external forces acting on the system.

Heaviside function:

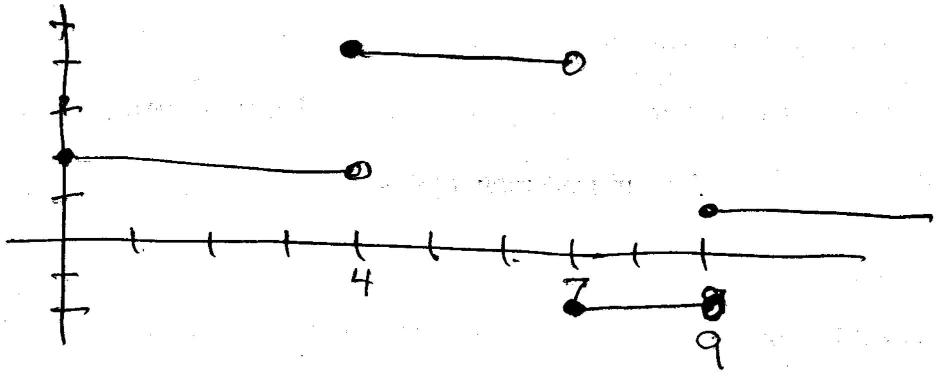
$$c \geq 0, \quad u_c(t) = \begin{cases} 0 & t<c \\ 1 & t \geq c \end{cases}$$



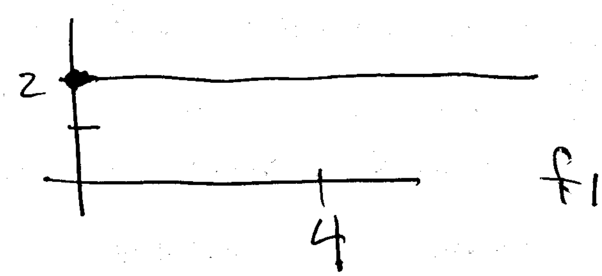
Fact: Any step function $f(t)$ on $[0, \infty)$ is a linear combination of Heaviside functions

eg

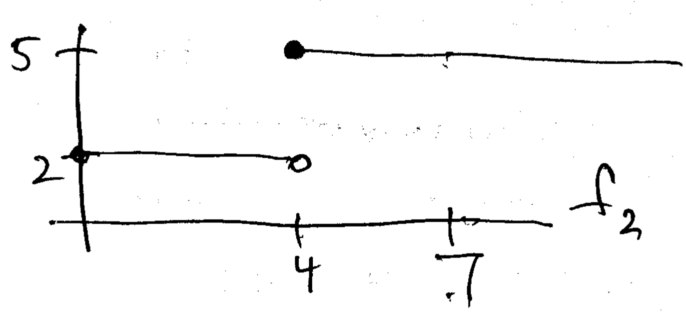
$$f(t) = \begin{cases} 2 & 0 \le t < 4 \\ 5 & 4 \le t < 7 \\ -1 & 7 \le t < 9 \\ 1 & t \ge 9 \end{cases}$$



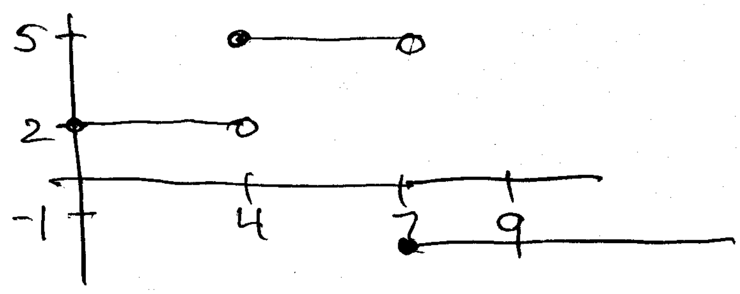
$$f_1(t) = 2u_0(t) = 2$$



$$f_2(t) = f_1(t) + \overset{(5-2)}{\overset{\downarrow}{3}}\, u_4(t)$$

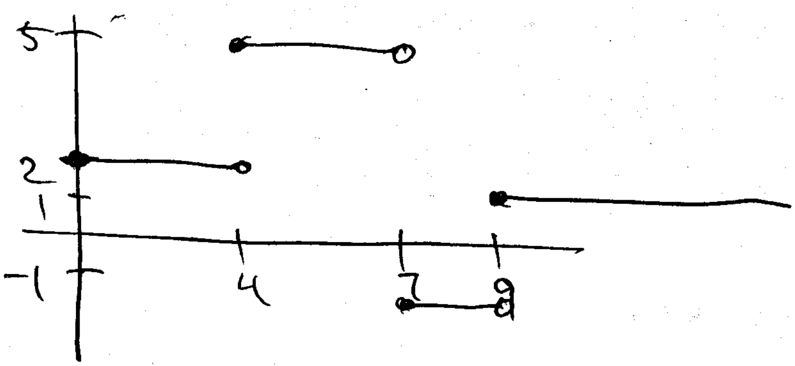


$$f_3(t) = f_2(t) \cancel{+} \overset{(-1-5)}{\overset{\swarrow}{-6}}\, u_7(t)$$



$$f_4(t) = f_3(t) + 2u_9(t)$$
$$= f(t)$$

$(1-(-1))$



$$f(t) = f_2(t) - 6u_7(t) + 2u_9(t)$$
$$= f_1(t) + 3u_4(t) - 6u_7(t) + 2u_9(t)$$
$$= 2 + 3u_4(t) - 6u_7(t) + 2u_9(t).$$