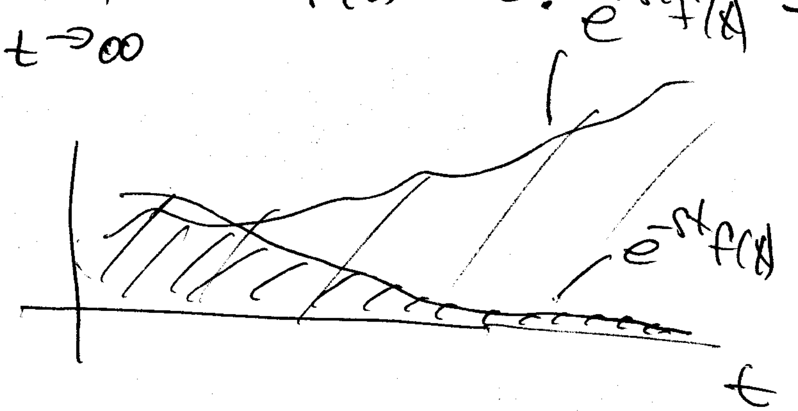
Quiz 9 – Section 3.6

Laplace Transforms.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt\,, \quad s \in \mathbb{R}.$$

Improper integral so it is defined only if $\lim_{t\to\infty} e^{-st} f(t) = 0.$ BUT THIS IS NOT ENOUGH!



We need it to go to zero FAST ENOUGH.

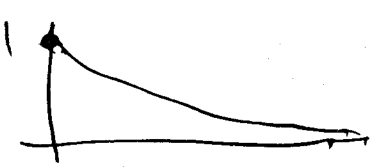
What is fast enough?

– Decay at infinity like $\frac{1}{t^p}$, $0<p\leq 1$ is not fast enough

$$\boxed{\int_1^\infty \frac{1}{t^p}\,dt = \infty \quad \text{if } 0<p\leq 1}$$

– Decay like $\frac{1}{t^p}$, $p>1$ is fast enough.

– Decay like e^{ct}, $c<0$ is fast enough

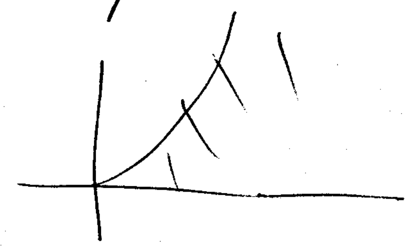


– Decay like e^{ct}, $c\geq 0$ is nowhere near fast enough.

($e^{ct} \to \infty$ as $t\to\infty$). $c>0$.

Bottom line: $\int_0^\infty e^{-st} f(t)\,dt$ will converge for some s and not others depending on f(t).

<u>e.g.</u> $f(t) = t^2$ $\quad \int_0^\infty t^2\,dt = \infty$



$$\int_0^\infty e^{-st} t^2\,dt = \lim_{A\to\infty} \int_0^A e^{-st} t^2\,dt$$

$s > 0$

$$= \lim_{A\to\infty} \left(\frac{1}{-s} t^2 e^{-st} \Big|_0^A - \frac{2}{-s} \int_0^A e^{-st} t\,dt \right)$$

$$= \lim_{A\to\infty} \underbrace{-\frac{1}{s} A^2 e^{-sA}}_{\to 0} + \frac{2}{s} \int_0^A e^{-st} t\,dt$$

$$= \frac{2}{s} \lim_{A\to\infty} \int_0^A e^{-st} t\,dt$$

$$= \frac{2}{s} \left(\cancelto{0}{\frac{1}{-s} t e^{-st} \Big|_0^A} - \frac{1}{-s} \int_0^A e^{-st}\,dt \right)$$

$\lim_{A\to\infty}$

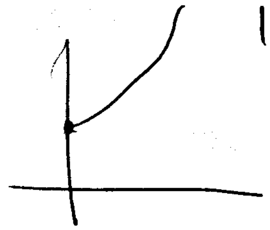
$$= \lim_{A\to\infty} \frac{2}{s^2} \left(-\frac{1}{s} e^{-st} \Big|_0^A \right)$$

$$= \lim_{A\to\infty} -\frac{2}{s^3} \left(\cancelto{0}{e^{-sA}} - 1 \right) = \frac{2}{s^3} = F(s).$$

<u>for $s > 0$</u>.

e.g. $f(t) = e^t$

Note: $e^t \not\to 0$ as $t \to \infty$



In fact $e^t \to \infty$ as $t \to \infty$.

But $\mathcal{L}\{e^t\}$ is still defined for some s.

$\mathcal{L}\{e^t\} =$

$$\int_0^\infty e^{-st} e^t \, dt = \lim_{A \to \infty} \int_0^A e^{-(s-1)t} \, dt$$

$s > 1$

$$= \lim_{A \to \infty} \frac{-1}{s-1} e^{-(s-1)t} \Big|_0^A = \lim_{A \to \infty} \frac{-1}{s-1} \left(\underbrace{e^{-(s-1)A}}_{0} - 1 \right)$$

since $s > 1$

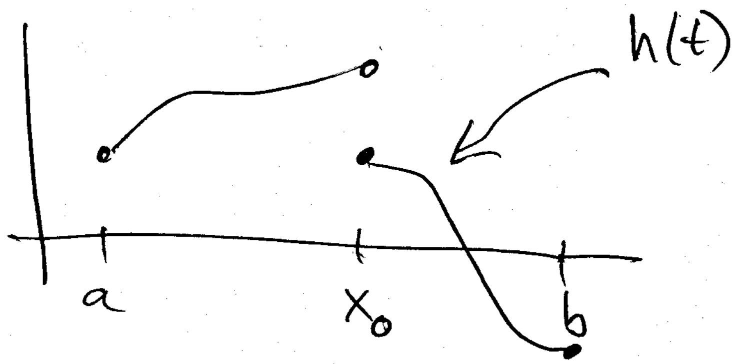
$$= \frac{1}{s-1} \text{ for } s > 1.$$

e.g. $f(t) = e^{-t}$ $\quad \mathcal{L}\{e^{-t}\} = \int_0^\infty e^{-st} e^{-t} \, dt$

$s > -1$

$$= \lim_{A \to \infty} \int_0^A e^{-(s+1)t} \, dt = \frac{1}{s+1}, \; s > -1$$

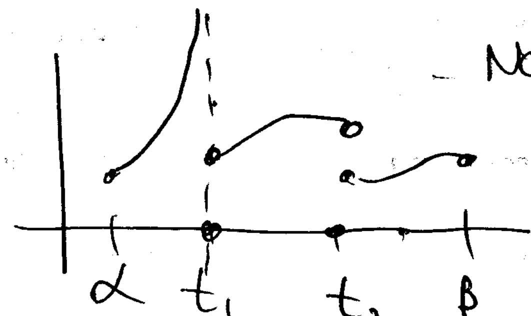
3) $\mathcal{L}\{f(t)\}$ is also defined if f has discontinuities



$$\int_a^b h(t)\,dt = \int_a^{x_0} h(t)\,dt + \int_{x_0}^b h(t)\,dt$$

<u>Def</u>: We say f is <u>piecewise continuous</u> on $[\alpha, \beta]$ if there are finitely many points $\alpha = t_0 < t_1 < t_2 < \cdots < t_n = \beta$ such that f is continuous on (t_{i-1}, t_i) $i = 1, \ldots, n$ <u>and</u> f has at worst a jump discontinuity at each t_i.

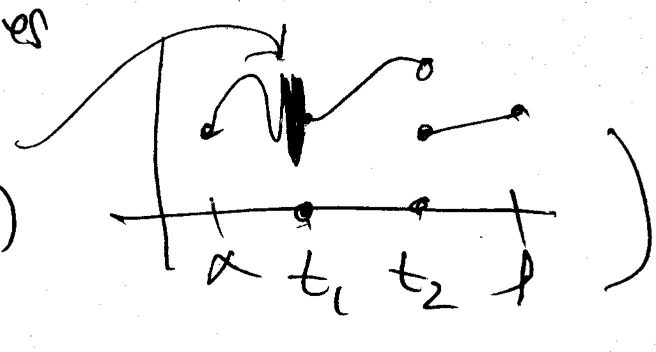
(i.e. no asymptotes



— No good.

no oscillatory discontinuities

like $\sin(\frac{1}{x})$



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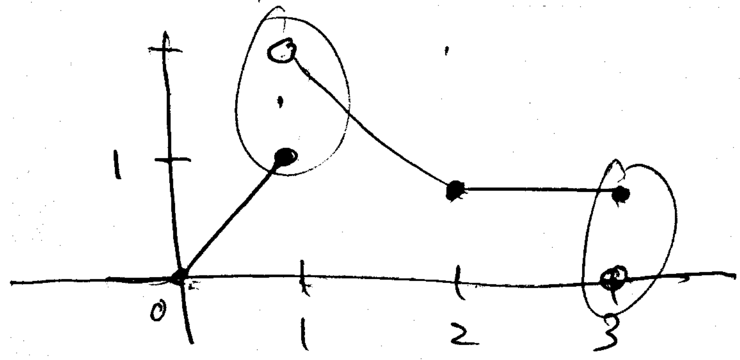
#4) $f(t) = \begin{cases} t & 0 \le t \le 1 \\ 3-t & 1 < t \le 2 \\ 1 & 2 < t \le 3 \\ 0 & \text{all other } t \end{cases}$



What is $\mathcal{L}\{f(t)\}$?

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^3 e^{-st} f(t)\,dt$$

$$= \int_0^1 t e^{-st}\,dt + \int_1^2 (3-t) e^{-st}\,dt + \int_2^3 e^{-st}\,dt$$

$$= \frac{e^{-st}}{(-s)^2}(-st-1)\Big|_0^1 + 3\int_1^2 e^{-st}\,dt - \int_1^2 t e^{-st}\,dt$$
$$+ \frac{1}{-s} e^{-st}\Big|_2^3$$

$$= \frac{e^{-s}}{s^2}(-s-1) + \frac{1}{s^2} + \frac{3}{-s} e^{-st}\Big|_1^2 - \frac{e^{-st}}{(-s)^2}(-st-1)\Big|_1^2$$
$$+ \frac{1}{-s}\left(e^{-3s} - e^{-2s}\right)$$

$$= \frac{-1}{s^2} e^{-s}(s+1) + \frac{1}{s^2} - \frac{3}{s}\left(e^{-2s} - e^{-s}\right) + \frac{e^{-2s}}{s^2}(2s+1)$$
$$- \frac{e^{-s}}{s^2}(s+1) - \frac{1}{s}\left(e^{-3s} - e^{-2s}\right).$$

<u>FACT:</u> $\mathcal{L}\{f(t)\}$ does not change if f is redefined at only a few points.

6-2 Solutions to IVPs.

Idea: Look at how $\mathcal{L}\{f'(t)\}$ is related to $\mathcal{L}\{f(t)\}$.

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)\,e^{-st}\,dt \qquad \begin{array}{ll} u = e^{-st} & dv = f'(t)\,dt \\ du = -se^{-st}\,dt & v = f(t) \end{array}$$

$$= \underbrace{e^{-st} f(t)\Big|_0^\infty}_{\cancel{\lim_{A\to\infty} e^{-sA} f(A)} \to 0 \; - f(0)} + s \underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{\mathcal{L}\{f(t)\}}$$

since $F(s)$ is defined.

$$= s\mathcal{L}\{f(t)\} - f(0).$$

$$\boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)}$$

$$\mathcal{L}\{f''(t)\} = s\,\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s\left(s\,\mathcal{L}\{f(t)\} - f(0)\right) - f'(0)$$

$$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

In general.

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - \underbrace{s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)}_{\text{polynomial that depends only on "initial conditions" of } f}.$$

polynomial that depends only on "initial conditions" of f.

eg 1. $y'' - y' - 2y = 0 \quad y(0) = 1,\ y'(0) = 0.$

$r^2 - r - 2 = 0$

$(r+1)(r-2) = 0$

$y_1 = e^{-t} \quad y_2 = e^{2t}$

$y = c_1 e^{-t} + c_2 e^{2t}$

$1 = c_1 + c_2$

$0 = -c_1 + 2c_2$

$1 = 3c_2 \quad c_2 = \frac{1}{3}$

$c_1 = \frac{2}{3}$

$y = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$

$\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{0\}$

$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$

$s^2\mathcal{L}\{y\} - sy(0) - y'(0)$

$\quad - (s\mathcal{L}\{y\} - y(0)) - 2\mathcal{L}\{y\} = 0$

$s^2\mathcal{L}\{y\} - s - s\mathcal{L}\{y\} + 1 - 2\mathcal{L}\{y\} = 0$

$\mathcal{L}\{y\}\underbrace{(s^2 - s - 2)}_{\text{char polynomial}} = s - 1$

$$\mathcal{L}\{y\} = \frac{s-1}{s^2 - s - 2}$$

Problem is: Find $y(t)$ with this as its Laplace transform.

Need to do partial fraction expansion of $\frac{s-1}{s^2-s-2}$.

$$\frac{s-1}{s^2-s-2} = \frac{s-1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} = \frac{A(s-2)+B(s+1)}{(s+1)(s-2)}$$

$$s-1 = A(s-2) + B(s+1) \qquad s=-1$$

$$-2 = -3A \qquad \underline{\underline{A = \frac{2}{3}}}$$

$$1 = 3B \qquad \underline{\underline{B = \frac{1}{3}}} \qquad s=2$$

$$\frac{s-1}{s^2-s-2} = \frac{2}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}$$

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-s-2}\right\} = \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$
$$= \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$$