Exam 1 - Tuesday 9/27

First order equations: $\frac{dy}{dt} = f(t,y)$

Direction field → long term behavior ($t \to \infty$)

no classifications

Simple models by first order linear equations

- falling objects (with air resistance)
- population growth (with predation)
- continuously compounded interest (with regular contributions/withdrawals)
- mixture models.

Solvable equations (IVP)

- separable
- linear (integrating factor)
- exact equations (integrating factor)

Existence of solutions

- Theorem 2.4.2. (verify conditions)
- determining the interval on which solutions exist (Thm 2.4.1 for linear eqns).

Continuing with 3.2 (Wronskian)

$L[y]=0$ $\qquad$ $L[\varphi]=\varphi''+p(t)\varphi'+q(t)\varphi$

Solve this

Know: 1) In general, these are hard.

2) Can solve $ay''+by'+cy=0$.

a) $ar^2+br+c=0$

$r=r_1$, $r=r_2$

b) if $r_1 \neq r_2$, and real then we write

$y_1=e^{r_1 t}$ $\quad$ $y_2=e^{r_2 t}$

c) General solution $y=c_1e^{r_1 t}+c_2e^{r_2 t}$.

d) initial conditions $\longrightarrow$ values of c_1, c_2.

3) Back to general case $L[y]=0$.

look for 2 solutions y_1, y_2.

Define $y=c_1y_1+c_2y_2$ as the general solution

Given initial conditions $y(t_0)=y_0$, $y'(t_0)=y_0'$

can we always solve for c_1, c_2?

That is, can we solve the system.

$$\begin{aligned} c_1y_1(t_0)+c_2y_2(t_0)&=y_0 \\ c_1y_1'(t_0)+c_2y_2'(t_0)&=y_0' \end{aligned} \quad ?$$

We get a unique solution if and only if

$$W(y_1,y_2)(t_0) = \det\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \neq 0$$

Define $W(y_1,y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$.

<u>Fact:</u> $W(t)$ satisfies

$$W'(t) + p(t)W(t) = 0 \quad \text{if } y_1, y_2 \text{ satisfy}$$

$$L[y_1] = L[y_2] = 0, \quad L[\varphi] = \varphi'' + p(t)\varphi' + q(t)\varphi$$

We know:

$$y_1'' + p(t)\,y_1' + q(t)\,y_1 = 0$$

$$y_2'' + p(t)\,y_2' + q(t)\,y_2 = 0$$

$$W(t) = y_1 y_2' - y_2 y_1'$$

$$W'(t) = y_1 y_2'' + \cancel{y_1' y_2'} - y_2 y_1'' - \cancel{y_2' y_1'}$$

$$= y_1 y_2'' - y_2 y_1''$$

$$y_1 y_2'' - y_2 y_1'' + p(t)\,y_1 y_2' - p(t)\,y_2 y_1'$$

$$= y_1\underbrace{(y_2'' + p(t)y_2' + q(t)y_2)}_{0} - y_2\underbrace{(y_1'' + p(t)y_1' + q(t)y_1)}_{0}$$

$= 0$. Hence $W(t) + p(t)W'(t) = 0$.

Important consequence:

$$W(t) = c\, e^{\int p(t)\,dt}.$$

This means: $W(t)$ is either always 0 (if $c=0$) or never zero (if $c \neq 0$) (as long as $p(t)$ is continuous)

Final conclusion:

If y_1, y_2 satisfy $L[y_1] = L[y_2] = 0$ and if $W(y_1, y_2) \neq 0$ at any point then all solutions to $L[y] = 0$ have the form $y = c_1 y_1 + c_2 y_2$. (This is related to Theorem 3.2.1).

Summary: To solve $y'' + p(t) y' + q(t) y = 0$

HARD PART → 1) Find 2 solutions y_1, y_2

2) Verify that $W(y_1, y_2) \neq 0$ at some point.

3) We say y_1, y_2 are a fundamental set of solutions. We also say they are linearly independent solutions.

4) Any solution has the form $y = c_1 y_1 + c_2 y_2$ for some c_1, c_2.

\# 32 $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

Find $W(y_1, y_2)$ where y_1, y_2 are solutions.
Find $p(t)$.

$$\underbrace{y'' - \frac{2x}{1-x^2}}_{p(t).} y' + \frac{\alpha(\alpha+1)}{1-x^2} y = 0.$$

$$p(t) = \frac{-2x}{1-x^2} \qquad \int p(x)\,dx = \int \frac{-2x}{1-x^2}\,dx$$

$$u = 1-x^2$$
$$du = -2x\,dx$$

$$= \int \frac{dy}{u} = \ln|u| = \ln|1-x^2|$$

$$W(y_1, y_2)(x) = c\,e^{\ln|1-x^2|} = c(1-x^2)$$

Note that $W(x) = 0$ if $x = \pm 1$ but $W(x) \neq 0$ on the interval $(-1, 1)$ which is where $p(t)$ is continuous. Also true on $(-\infty, -1)$ and $(1, \infty)$.

3.3 Complex Roots.

Back to $ay'' + by' + cy = 0$.

e.g. $y'' + y' + 9.25y = 0$

$$r^2 + r + 9.25 = 0$$

$$r = \frac{-1 \pm \sqrt{1 - 4(9.25)}}{2} = \frac{-1 \pm \sqrt{-36}}{2} = \frac{-1 \pm 6i}{2}$$

$$= -\tfrac{1}{2} \pm 3i \qquad r_1 = -\tfrac{1}{2} + 3i, \quad r_2 = -\tfrac{1}{2} - 3i$$

complex conjugates.

Solutions: $y_1 = e^{(-\frac{1}{2}+3i)t}$ $\qquad y_2 = e^{(-\frac{1}{2}-3i)t}$

$$y_1 = e^{(-\frac{1}{2}+3i)t} = e^{-\frac{1}{2}t + 3ti} = e^{-t/2}\, e^{3ti}$$

what does this mean?

$$y_2 = e^{(-\frac{1}{2}-3i)t} = e^{-\frac{1}{2}t} e^{-3ti} = e^{-t/2}\, e^{-3ti}$$

Euler's formula: What is e^{ix}?

$$e^t = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \quad \text{(Taylor series)}$$

Idea: extend to imaginary t.

Define $e^{ix} = \sum_{n=0}^{\infty} \frac{1}{n!}(ix)^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n (i)^n$

$i \quad i^2=-1 \quad i^3 = i^2 \cdot i = -i \quad i^4 = 1 \quad i^5 = i$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} (i)^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1} (i)^{2k+1}$$

$i^{2k+1} = i^{2k} \cdot i = (-1)^k i$

$$= \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}}_{\cos(x)} + i \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}}_{\sin(x)}$$

Taylor series

$= \cos(x) + i\sin(x).$

Euler's formula: $\boxed{e^{ix} = \cos(x) + i\sin(x)}$

Back to example:

$y_1 = e^{-t/2} e^{3ti} = e^{-t/2}(\cos(3t) + i\sin(3t))$ ✓

$y_2 = e^{-t/2} e^{-3ti} = e^{-t/2}(\cos(-3t) + i\sin(-3t))$

$= e^{-t/2}(\cos(3t) - i\sin(3t))$ ✓

Check $W(y_1, y_2)(t)$.

Show you something: $y_1 = e^{r_1 t}$ $y_2 = e^{r_2 t}$

r_1, r_2 real or complex

$$W(e^{r_1 t}, e^{r_2 t}) = e^{r_1 t} \cdot r_2 e^{r_2 t} - e^{r_2 t} r_1 e^{r_1 t}$$
$$= e^{(r_1 + r_2)t}(r_2 - r_1)$$

So $W = 0$ if and only if $r_1 = r_2$.

Since $-\frac{1}{2} + 3i \neq -\frac{1}{2} - 3i$, $W(y_1, y_2) \neq 0$.

Any solution has the form

$$y = c_1 e^{-t/2}(\cos(3t) + i\sin(3t)) + c_2 e^{-t/2}(\cos(3t) - i\sin(3t)).$$

Is there a simpler fundamental set?

Yes! I can take

$$y_1 = e^{-t/2}\cos(3t) \quad y_2 = e^{-t/2}\sin(3t)$$

these solve the equation <u>and</u>

$$W(e^{-t/2}\cos 3t, e^{-t/2}\sin 3t)$$
$$= e^{-t/2}\cos 3t\left(3e^{-t/2}\cos 3t - \frac{1}{2}e^{-t/2}\sin 3t\right)$$
$$- e^{-t/2}\sin 3t\left(-3e^{-t/2}\sin 3t - \frac{1}{2}e^{-t/2}\cos 3t\right)$$

$$= 3e^{-t}(\overset{1}{\cancel{\cos^2 3t + \sin^2 3t}})$$

$$+ \left(\overset{0}{\cancel{-\frac{1}{2}e^{-t}\cos 3t \sin 3t + \frac{1}{2}e^{-t}\cos 3t \sin 3t}}\right)$$

$$= 3e^{-t} \neq 0.$$

$\therefore$ Any solution has the form

$$y = c_1 e^{-t/2}\cos 3t + c_2 e^{-t/2}\sin 3t$$