Quiz 13: Sections 6.4, 6.5.

Systems of 1st order equations

nth order equation: $y^{(n)} = F(t, y, y', \dots, y^{(n-1)})$
(general)

Any such equation can be written as a 1st order system.

$$x_1 = y \qquad \boxed{x_1' = y' = x_2}$$
$$x_2 = y' \qquad x_2' = x_3$$
$$x_3 = y'' \qquad x_3' = x_4$$
$$\vdots \qquad \vdots$$
$$x_n = y^{(n-1)} \qquad x_n' = y^{(n)} = F(t, x_1, x_2, \dots, x_n)$$

Most general system:

$$x_1' = F_1(t, x_1, x_2, \dots, x_n)$$
$$x_2' = F_2(t, x_1, x_2, \dots, x_n)$$
$$\vdots$$
$$x_n' = F_n(t, x_1, \dots, x_n)$$

Initial conditions look like:

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \dots, x_n(t_0) = x_n^0.$$

A <u>linear system</u> means all the F_i are linear functions, i.e.

$$x_1'(t) = p_{11}(t)x_1(t) + p_{12}(t)x_2(t) + \cdots + p_{1n}(t)x_n(t) + g_1(t)$$
$$x_2'(t) = p_{21}(t)x_1(t) + p_{22}(t)x_2(t) + \cdots + p_{2n}(t)x_n(t) + g_2(t)$$
$$\vdots$$
$$x_n'(t) = p_{n1}(t)x_1(t) + p_{n2}(t)x_2(t) + \cdots + p_{nn}(t)x_n(t) + g_n(t)$$

The system is homogeneous if all the g_i's are <u>zero</u>.

<u>e.g.</u> #6) $u'' + p(t)u' + q(t)u = g(t)$ $\quad u(0) = u_0$

$u'(0) = u_0'$

$$x_1 = u \quad \longrightarrow \quad x_1' = x_2$$

$$x_2 = u' \qquad x_2' = -p(t)x_2 - q(t)x_1 + g(t)$$

$$\boxed{x_2' = u'' = -p(t)u' - q(t)u + g(t)}$$

linear system.

eg #7) $x_1' = -2x_1 + x_2$

$x_2' = x_1 - 2x_2$

$x_1'' = -2x_1' + x_2' = -2x_1' + x_1 - 2x_2$

$x_2 = x_1' + 2x_1$

$= -2x_1' + x_1 - 2x_1' - 4x_1$

$= -4x_1' - 3x_1$

So system becomes:

$x_1'' + 4x_1' + 3x_1 = 0$

<u>Solve:</u> $r^2 + 4r + 3 = 0$

$(r+1)(r+3) = 0$

$r = -1 \quad r = -3$

$x_1 = c_1 e^{-3t} + c_2 e^{-t}$

$x_2 = x_1' + 2x_1$

$= -3c_1 e^{-3t} - c_2 e^{-t}$

$+ 2c_1 e^{-3t} + 2c_2 e^{-t}$

$= -c_1 e^{-3t} + c_2 e^{-t}$

General solution:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-3t} + c_2 e^{-t} \\ -c_1 e^{-3t} + c_2 e^{-t} \end{pmatrix}$$

$$= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

these vectors are <u>special</u>

Put in some initial conditions

$x_1(0) = 2 \quad x_2(0) = 3$

To convert initial conditions

$x_1'(0) = -2x_1(0) + x_2(0) = -4+3 = -1$

$x_1(t) = c_1 e^{-3t} + c_2 e^{-t} \rightarrow 2 = c_1 + c_2$

$x_1'(t) = -3c_1 e^{-3t} - c_2 e^{-t} \rightarrow -1 = -3c_1 - c_2$

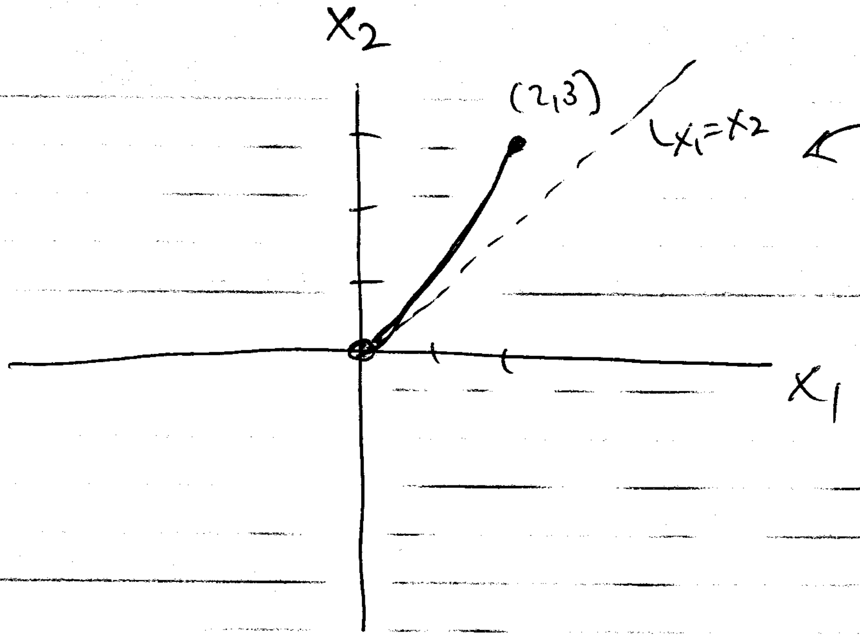
$1 = -2c_1$

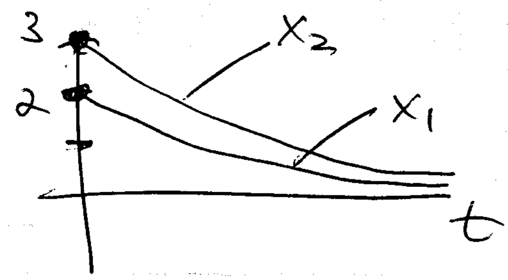
$c_1 = -\frac{1}{2} \quad c_2 = \frac{5}{2}$

$x_1(t) = -\frac{1}{2}e^{-3t} + \frac{5}{2}e^{-t}$

Solution to system: $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + \frac{5}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$

$= \begin{pmatrix} -\frac{1}{2}e^{-3t} + \frac{5}{2}e^{-t} \\ \frac{1}{2}e^{-3t} + \frac{5}{2}e^{-t} \end{pmatrix}$





$$\frac{dx_2}{dx_1} = \frac{dx_2/dt}{dx_1/dt} = \frac{-\frac{1}{2}e^{-3t}+\frac{5}{2}e^{-t}}{\frac{1}{2}e^{-3t}+\frac{5}{2}e^{-t}}$$

≈ 1 as $t \to \infty$.

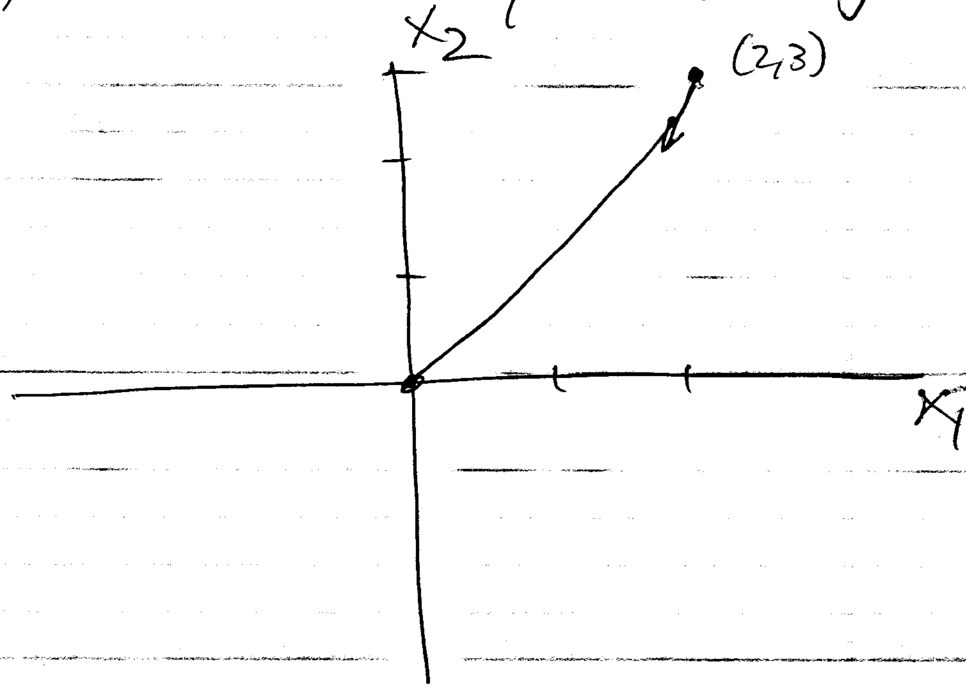
Another way to get same curve qualitatively.

$$x_1' = -2x_1 + x_2$$
$$x_2' = x_1 - 2x_2$$

$(x_1, x_2) = (2, 3)$

$x_1' = -4 + 3 = -1$

$x_2' = 2 - 6 = -4$



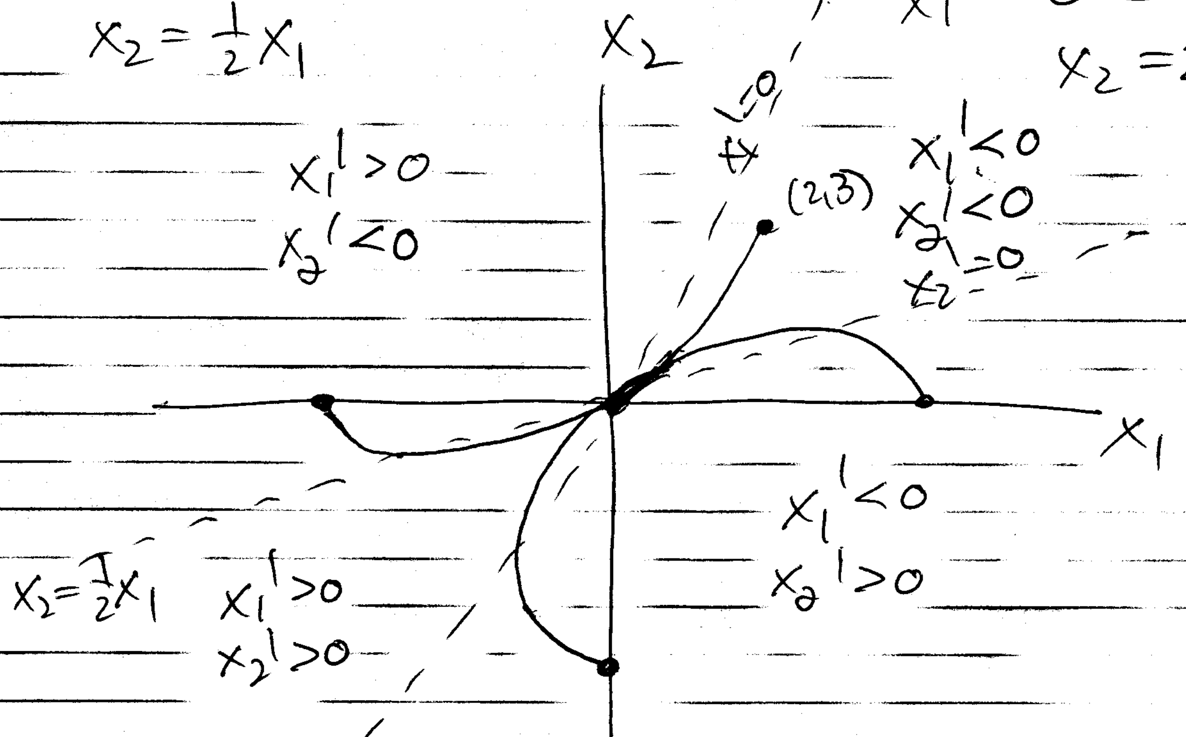
More generally:

$x_2' = 0 = x_1 - 2x_2$

$x_2 = \frac{1}{2}x_1$

$x_1' = 0 = -2x_1 + x_2$

$x_2 = 2x_1$



$x_2 = 2x_1$

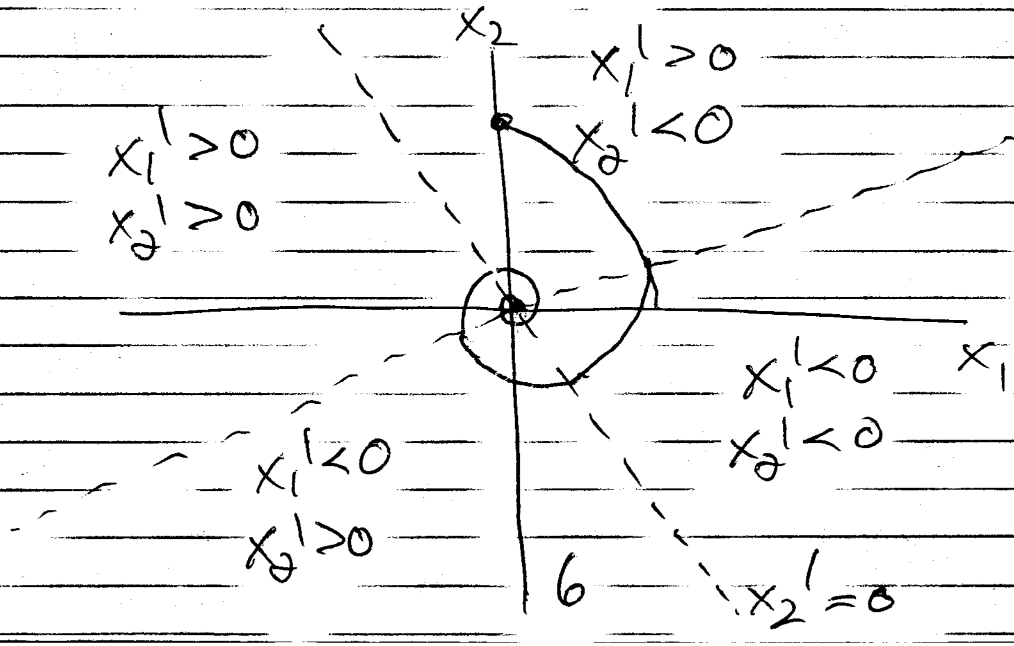
No matter where I start,

$(x_1(t), x_2(t)) \to (0,0)$

as $t \to \infty$.

<u>eg</u> #12) $x_1' = -\frac{1}{2}x_1 + 2x_2 \longrightarrow x_1' = 0 \to x_2 = \frac{1}{4}x_1$

$x_2' = -2x_1 - \frac{1}{2}x_2 \longrightarrow x_2 = -4x_1$



decaying oscillations

7.2 Review of Matrices

$$x_1' = p_{11}x_1 + p_{12}x_2 + \cdots + p_{1n}x_n$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$x_n' = p_{n1}x_1 + p_{n2}x_2 + \cdots + p_{nn}x_n$$

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ & & \vdots & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\vec{x}'(t) = P(t)\,\vec{x}(t)$$

For homogeneous equations with const. coefficient we write

$$\vec{x}'(t) = \underset{\approx}{A}\,\vec{x}(t) \qquad A = \text{constant } n \times n \text{ matrix}$$

Adding + Subtracting matrices

eg #6) $A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 3 \\ 1 & 0 & 2 \end{pmatrix}$

$$A - 2B = \begin{pmatrix} 1-2(2) & -2-2(1) & 0-2(-1) \\ 3-2(-2) & 2-2(3) & -1-2(3) \\ -2-2(1) & 0-2(0) & 3-2(2) \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -4 & 2 \\ 7 & -4 & -7 \\ -4 & 0 & -1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 3 & -2 \\ -2 & 2 & 0 \\ 0 & -1 & 3 \end{pmatrix}$$

$$AB = \begin{array}{c} \begin{matrix} 2 & -2 & 1 \end{matrix} \\ \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{pmatrix} \end{array} \begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 3 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 6 & -5 & -7 \\ 1 & 9 & 1 \\ -1 & -2 & 8 \end{pmatrix}$$

In general $AB \neq BA$

Finding inverses or determine if they exist

eg #10 $\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}$ $AA^{-1} = I$

Determinant: $\det\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix} = (1)(3) - (4)(-2)$
$= 11$

$$\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{11} & \frac{-4}{11} \\ \frac{2}{11} & \frac{1}{11} \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \cdot \frac{1}{ad-bc}$$

ck: $\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 2 & 1 \end{pmatrix} \cdot \frac{1}{11} = \begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix} \cdot \frac{1}{11} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$