nth order linear equations

1. General theory of solving.
2. Can solve homogeneous equations w/const. coeffs.
3. Method of undetermined coeffs:
   - can find a particular solution for certain non-homogeneous terms ($g(t)$), and if we have const. coeffs.
4. Method of Variation of Parameters.
   - allows us to find a particular solution given a fund. system of homogeneous solutions.

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e.g. $y''' - y' = t$

Fundamental system:

$$r^3 - r = 0$$
$$r(r^2-1) = 0$$
$$r(r+1)(r-1) = 0$$
$$r=0,\ r=-1,\ r=1$$
$$\boxed{y_1 = 1,\ y_2 = e^{-t},\ y_3 = e^{t}}$$

Look for particular solution of the form:

$$Y(t) = u_1(t) + u_2(t)e^{-t} + u_3(t)e^{t}$$
$$Y'(t) = u_1'(t) + u_2'(t)e^{-t} - u_2(t)e^{-t} + u_3'(t)e^{t} + u_3(t)e^{t}$$

Assume: $u_1' + u_2' e^{-t} + u_3' e^{t} = 0$

$$\therefore Y'(t) = -u_2 e^{-t} + u_3 e^{t}$$

$$y''(t) = u_2 e^{-t} - u_2' e^{-t} + u_3 e^t + u_3' e^t$$

Assume: $-u_2' e^{-t} + u_3' e^t = 0$

$$y''(t) = u_2 e^{-t} + u_3 e^t$$

$$y'''(t) = -u_2 e^{-t} + u_2' e^{-t} + u_3 e^t + u_3' e^t$$

$$= (-u_2 e^{-t} + u_3 e^t) + (u_2' e^{-t} + u_3' e^t)$$

always disappear when plugged into orig. equation

Plug into original:

$$y''' - y' = \cancel{(-u_2 e^{-t} + u_3 e^t)} + (u_2' e^{-t} + u_3' e^t)$$

$$- \cancel{(-u_2 e^{-t} + u_3 e^t)} = t$$

Solve the system:

$$u_1' + u_2' e^{-t} + u_3' e^t = 0$$

$$-u_2' e^{-t} + u_3' e^t = 0$$

$$u_2' e^{-t} + u_3' e^t = t$$

$$2u_3' e^t = t$$

$$u_3' = \frac{1}{2} t e^{-t}$$

$$u_3 = \int \frac{1}{2} t e^{-t} dt = \frac{1}{2}\left(-te^{-t} + \int e^{-t} dt\right)$$

$$= -\frac{1}{2} t e^{-t} - \frac{1}{2} e^{-t} \text{ //}$$

$$u_2' e^{-t} + \left(\tfrac{1}{2} t \cancel{e^{-t}}\right) \cancel{e^{t}} = t$$

$$u_2' e^{-t} = \tfrac{1}{2} t$$

$$u_2' = \tfrac{1}{2} t e^{t} \longrightarrow u_2 = \tfrac{1}{2}\left(t e^{t} - \int e^{t} dt\right)$$

$$= \tfrac{1}{2} t e^{t} - \tfrac{1}{2} e^{t} \,//$$

$$u_1' + \tfrac{1}{2} t + \tfrac{1}{2} t = 0$$

$$u_1' = -t \longrightarrow u_1 = -\tfrac{1}{2} t^2 \,//$$

$$Y(t) = -\tfrac{1}{2} t^2 + \cancel{\tfrac{1}{2} t} - \tfrac{1}{2} + \left(-\cancel{\tfrac{1}{2} t} - \tfrac{1}{2}\right)$$

$$= -\tfrac{1}{2} t^2 - 1$$

Note: Could also have gotten this using undetermined coefficients.

Our system was:

$$\begin{bmatrix} 1 & e^{-t} & e^{t} \\ 0 & -e^{-t} & e^{t} \\ 0 & e^{-t} & e^{t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

In general, for n^{th} order equation:

1. Look for $Y(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x)$

2. Solve system

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(x) \end{bmatrix}$$

e.g. #6 (just set up).

$$y^{(4)} + 2y'' + y = \sin(t).$$

1) Solve homogeneous equation

$r^4 + 2r^2 + 1 = 0$ $\quad y_1 = \cos t$

$(r^2+1)^2 = 0$ $\quad y_2 = \sin t$

$r = \pm i$ $\quad y_3 = t\cos t$

$y_4 = t\sin t$

Let's set up the system:

$$Y(t) = u_1(t)\cos t + u_2(t)\sin t + u_3(t)t\cos t + u_4(t)t\sin(t)$$

Solve:

$$\begin{bmatrix} \cos t & \sin t & t\cos t & t\sin t \\ -\sin t & \cos t & -t\sin t+\cos t & t\cos t+\sin t \\ -\cos t & -\sin t & -t\cos t-2\sin t & -t\sin t+2\cos t \\ \sin t & -\cos t & t\sin t-3\cos t & -t\cos t-3\sin t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin t \end{bmatrix}$$

Cramer's Rule:

6.1 Laplace Transform

Def: Given f, the Laplace transform of f, denoted $\mathcal{L}\{f(t)\} = F(s)$ is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt \quad \text{for } s \in \mathbb{R}.$$

Basic observations:

1) $\mathcal{L}$ is an integral transform (or integral operator). We have encountered a differential operator already, called L.

i.e. $L[y] = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y' + p_n y$.

f is a function; $\mathcal{L}\{f(t)\}$ is also a function.

2) For each $s \in \mathbb{R}$, $F(s)$ is an improper integral.

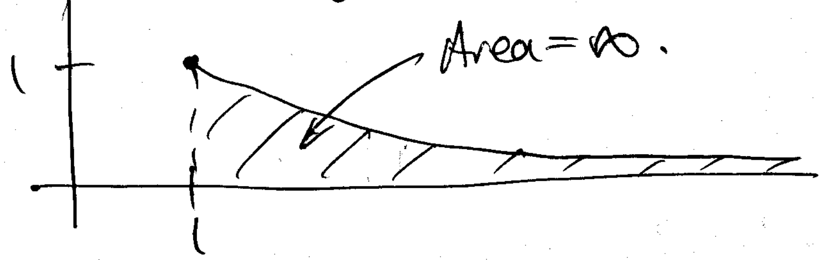
Recall: $$\int_a^{\infty} h(t)\,dt = \lim_{A \to \infty} \int_a^A h(t)\,dt$$

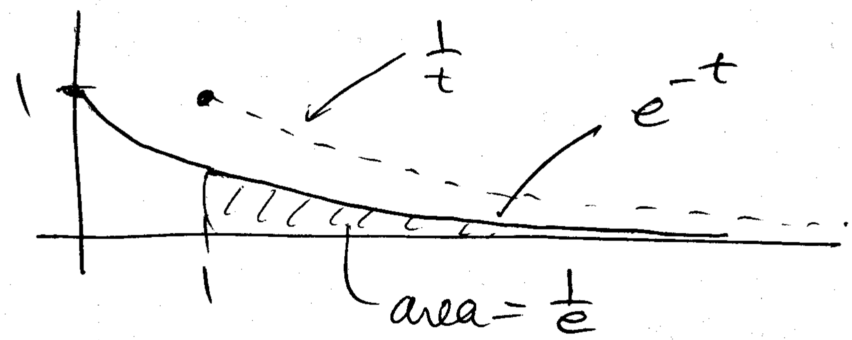
a) The limit may not exist.

e.g. $$\int_1^{\infty} \frac{1}{t}\,dt = \lim_{A\to\infty} \int_1^A \frac{1}{t}\,dt = \lim_{A\to\infty} \ln(t)\Big|_1^A$$

$$= \lim_{A\to\infty} \ln A = \infty$$

$f(t)=\frac{1}{t}$





$$\int_1^\infty e^{-t}\,dt = \lim_{A\to\infty}\int_1^A e^{-t}\,dt$$

$$= \lim_{A\to\infty} -e^{-t}\Big|_1^A = \lim_{A\to\infty}\left(-e^{-A}+e^{-1}\right) = e^{-1}$$

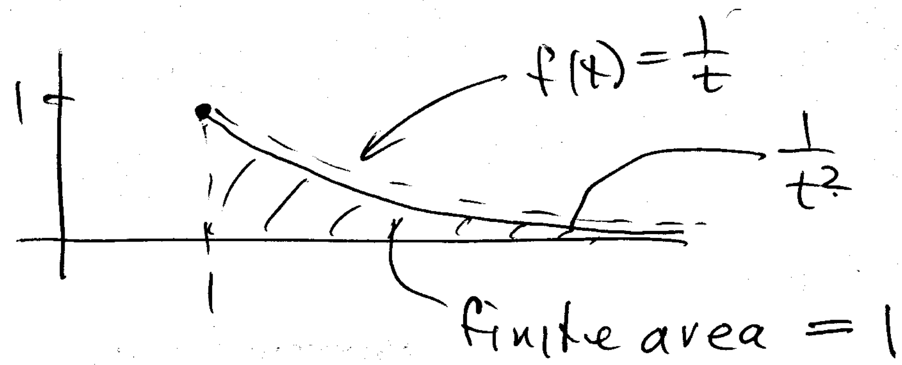
For the integral to converge, we must have $h(t)\to 0$ as $t\to\infty$ fast enough

e.g. $\displaystyle\int_1^\infty \frac{1}{t^p}\,dt = \lim_{A\to\infty}\int_1^A \frac{1}{t^p}\,dt$

$p>0,\ p\neq 1$

$$= \lim_{A\to\infty} \frac{1}{t^{p-1}}\cdot\frac{1}{1-p}\Big|_1^A = \lim_{A\to\infty}\left(\frac{1}{A^{p-1}}\cdot\frac{1}{1-p} - \frac{1}{1-p}\right)$$

$$= \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } 0<p<1 \end{cases}$$

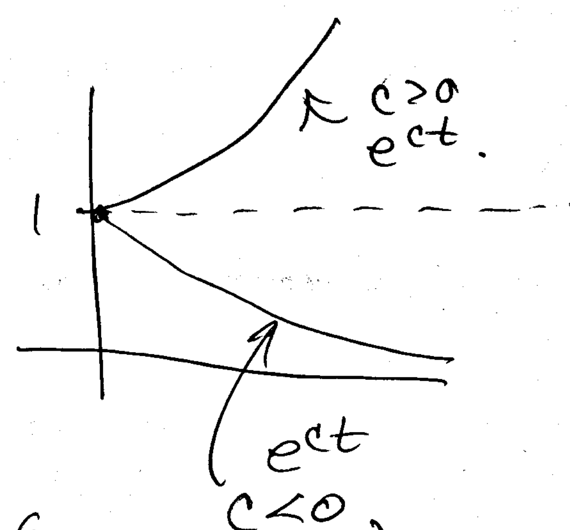


$$\int_0^\infty e^{ct}\,dt = \lim_{A\to\infty} \int_0^A e^{ct}\,dt$$

$$= \lim_{A\to\infty} \frac{1}{c} e^{ct}\Big|_0^A = \lim_{A\to\infty} \left(\frac{1}{c} e^{Ac} - \frac{1}{c}\right)$$

~~If $c<0$~~

$$= \begin{cases} -\frac{1}{c} & \text{if } c<0 \\ \infty & \text{if } c>0. \end{cases}$$



3) $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ will (in general) converge for some values of s and not for others.

e.g. $\mathcal{L}\{t^2\} = \int_0^\infty e^{-st} t^2\,dt$, $s>0$. ← converges for all $s>0$.

$\boxed{\text{Exponential decay always beats polynomial growth.}}$