Quiz 8 - Undetermined Coeffs (3.5).

nth order linear equations

$$\underbrace{y^{(n)} + p_1 y^{(n-1)} + p_2 y^{(n-2)} + \cdots + p_{n-1} y' + p_n y}_{L[y]} = g.$$

4.2 Homogeneous Equations with constant coefficients.

e.g. $y^{(4)} - y = 0$ $\qquad y_1 = \cos(t)$

$r^4 - 1 = 0$ $\qquad y_2 = \sin(t)$

$(r^2+1)(r^2-1) = 0$ $\qquad y_3 = e^t$

$(r^2+1)(r-1)(r+1) = 0$ $\qquad y_4 = e^{-t}$

$r = \pm i,\ r = 1,\ r = -1$

$$y = c_1\cos(t) + c_2\sin(t) + c_3 e^t + c_4 e^{-t}.$$

#12 $y''' - 3y'' + 3y' - y = 0$

$r^3 - 3r^2 + 3r - 1 = 0$ $\qquad r = 1$ works

$$\begin{array}{r|l} & r^2 - 2r + 1 \\ r-1 & r^3 - 3r^2 + 3r - 1 \\ & -r^3 + r^2 \\ \hline & -2r^2 + 3r \\ & +2r^2 - 2r \\ \hline & r - 1 \end{array}$$

$(r-1)\underbrace{(r^2-2r+1)}_{(r-1)(r-1)} = 0$

$(r-1)^3 = 0$ $\qquad r = 1$

$y_1 = e^t, \quad y_2 = te^t, \quad y_3 = t^2 e^t$

$y = c_1 e^t + c_2 te^t + c_3 t^2 e^t$

#31) $y^{(4)} - 4y''' + 4y'' = 0$ $\quad y(1) = -1 \quad y'(1) = 2$

$y''(1) = 0 \quad y'''(1) = 0$

$r^4 - 4r^3 + 4r^2 = 0$

$r^2(r^2 - 4r + 4) = 0$

$r^2(r-2)(r-2) = 0$

$r^2(r-2)^2 = 0$

$r = 0 \quad r = 2$

$e^{0t} = 1$

$te^{0t} = t$

$y_1 = 1$

$y_2 = t$

$y_3 = e^{2t}$

$y_4 = te^{2t}$

$\boxed{y = c_1 + tc_2 + c_3 e^{2t} + c_4 te^{2t}}$

$y' = c_2 + 2c_3 e^{2t} + c_4(2te^{2t} + e^{2t})$

$y'' = 4c_3 e^{2t} + c_4(4te^{2t} + 4e^{2t})$

$y''' = 8c_3 e^{2t} + c_4(8te^{2t} + 12e^{2t})$

$-1 = c_1 + c_2 + c_3 e^2 + c_4 e^2 \rightarrow -1 = c_1 + c_2 \rightarrow \underline{c_1 = -3}$

$2 = c_2 + 2c_3 e^2 + 3c_4 e^2 \longrightarrow \underline{2 = c_2}$

$0 = 4c_3 e^2 + 8c_4 e^2$

$0 = 8c_3 e^2 + 20c_4 e^2$

$\longrightarrow c_3 = 0$

$\underline{c_4 = 0}$

$y = -3 + 2t$ //

4.3 Undetermined Coefficients

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y' + p_n y = g$$

Non-homogeneous equation.

Idea: If $g(t)$ has certain particular forms we can guess at the form of a particular solution $Y(t)$ and solve for unknown coefficients. We do not have to have already solved the homogeneous equation.

#2) $y^{(4)} - y = 3t + \cos t$

$$Y(t) = \cancel{A} + Bt + C\cos t + D\sin t$$

$$Y'(t) = B - C\sin t + D\cos t$$

$$Y''(t) = -C\cos t - D\sin t$$

$$Y'''(t) = C\sin t - D\cos t$$

$$Y^{(4)}(t) = C\cos t + D\sin t$$

$$C\cancel{\cos t} + \cancel{D\sin t} - Bt - \cancel{C\cos t} - \cancel{D\sin t} = 3t + \cos t$$

$\boxed{B = -3}$ $\cos t$, $\sin t$ solve homogeneous equation.

$$Y(t) = At\cos t + Bt\sin t$$

$$Y'(t) = -At\sin t + A\cos t + Bt\cos t + B\sin t$$

$$= (-At+B)\sin t + (A+Bt)\cos t$$

$$Y'' = (-At+B)\cos t - A\sin t + (-A-Bt)\sin t + B\cos t$$

$$= (-At+2B)\cos t + (-2A-Bt)\sin t$$

$$Y''' = (At-2B)\sin t - A\cos t + (-2A-Bt)\cos t - B\sin t$$

$$= (At-3B)\sin t + (-3A-Bt)\cos t$$

$$Y^{(4)} = (At-3B)\cos t + A\sin t + (3A+Bt)\sin t - B\cos t$$

$$= (At-4B)\cos t + (4A+Bt)\sin t.$$

$$(\cancel{At}-4B)\cos t + (4A+\cancel{Bt})\sin t - \cancel{At\cos t} - \cancel{Bt\sin t} = \cos t$$

$$-4B\cos t + 4A\sin t = \cos t$$

$$A = 0 \quad B = -\frac{1}{4}.$$

$$\therefore Y(t) = -3t - \frac{1}{4}t\sin t. //$$

For general solution, solve $y^{(4)} - y = 0$

$r^4 - 1 = 0$ $\quad r = \pm i$ $\quad y_1 = \cos t$

$(r^2+1)(r^2-1) = 0$ $\quad r = 1$ $\quad y_2 = \sin t$

$(r^2+1)(r-1)(r+1)$ $\quad r = -1$ $\quad y_3 = e^t, \; y_4 = e^{-t}$

General solution:

$$y = c_1 \cos t + c_2 \sin t + c_3 e^{t} + c_4 e^{-t} - 3t - \frac{1}{4} t \sin t.$$

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e.g. #9. $y''' + 4y' = t \qquad y(0) = y'(0) = 0 \quad y''(0) = 1$

Solve homogeneous eqn first:

$$r^3 + 4r = 0 \qquad y_1 = 1$$

$$r(r^2+4) = 0 \qquad y_2 = \cos(2t)$$

$$r = 0 \quad r = \pm 2i \qquad y_3 = \sin(2t)$$

Particular solution. $Y(t) = At$

$$Y' = A, \ Y'' = 0, \ Y''' = 0$$

$$0 + 4A = t \quad \text{uh oh!}$$

$$Y(t) = At^2 \quad Y'(t) = 2At, \ Y''(t) = 2A \quad Y'''(t) = 0$$

$$0 + 8At = t \longrightarrow A = \frac{1}{8} \quad \boxed{Y(t) = \frac{1}{8} t^2}$$

General solution:

$$y = c_1 + c_2 \cos(2t) + c_3 \sin(2t) + \frac{1}{8} t^2 \quad //$$

Initial conditions:

$$y' = -2c_2 \sin(2t) + 2c_3 \cos(2t) + \frac{1}{4} t$$

$$y'' = -4c_2 \cos(2t) - 4c_3 \sin(2t) + \frac{1}{4}.$$

$0 = c_1 + c_2 \rightarrow 0 = c_1 - \frac{3}{16} \rightarrow \underline{c_1 = \frac{3}{16}}$

$0 = 2c_3 \longrightarrow \underline{c_3 = 0}$

$1 = -4c_2 + \frac{1}{4} \rightarrow \frac{3}{4} = -4c_2 \rightarrow \underline{c_2 = \frac{-3}{16}}$

$y = \frac{3}{16} - \frac{3}{16}\cos(2t) + \frac{1}{8}t^2$ //

#14) $y''' - y' = te^{-t} + 2\cos t$

Find an appropriate form of $Y(t)$.

Solve homog eqn:

$r^3 - r = 0$

$r(r^2 - 1) = 0$

$r = 0, r = 1, r = -1$

$y_1 = 1$

$y_2 = e^t$

$y_3 = e^{-t}$

$Y(t) = At^2e^{-t} + Bte^{-t} + C\cos(t) + D\sin(t)$.

$Bte^{-t} = Y$

$Y' = -Bte^{-t} + Be^{-t}$

$Y'' = +Bte^{-t} - 2Be^{-t}$

$Y''' = -Bte^{-t} + 3Be^{-t}$

$-\cancel{Bte^{-t}} + 3Be^{-t} + \cancel{Bte^{-t}} - Be^{-t} = te^{-t}$

$2Be^{-t} = te^{-t}$

can't solve this