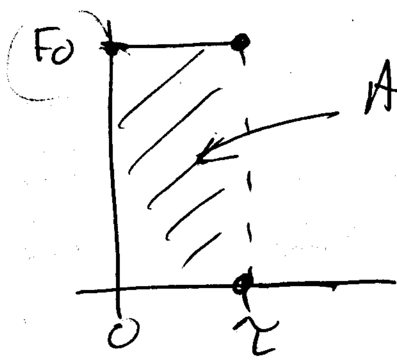
Impulse Functions (as forcing functions)

Idea:



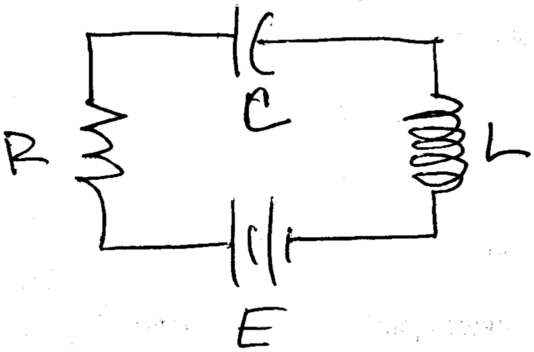
Area = "impulse" of the force
= change in velocity
of unit mass.

Define $d_\tau(t) = \begin{cases} 1/\tau & \text{if } 0 \le t < \tau \\ 0 & \text{if } t \ge \tau. \end{cases}$

then $\delta(t) = \lim_{\tau \to 0} d_\tau(t)$. Then 1) $\delta(t) = 0$ if $t \neq 0$
2) $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

Models: 1) For mass-spring system, $\delta(t)$ models hitting the mass with a hammer.

2) For electronic circuits:



$LQ'' + RQ' + \frac{1}{C}Q = E(t)$

Q = charge on capacitor.

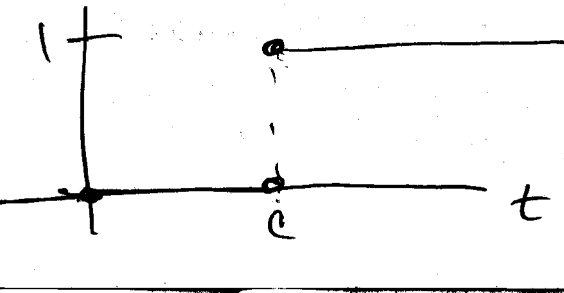
Since $I = \frac{dQ}{dt}$ we also have.

current

$LI'' + RI' + \frac{1}{C}I = E'(t)$

If

E(t):



(turning on a switch)

how do we model $E'(t)$?

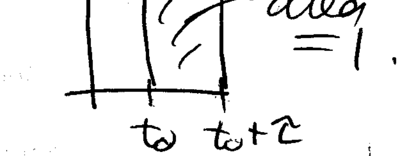
1) $E'(t)=0$ if $t\neq c$. 2) $\displaystyle\int_{-\infty}^{\infty} E'(t)\,dt = E(\infty)-E(-\infty)$

$= 1-0 = 1$

So $E' = \delta(t-c)$ makes sense.

Laplace transform.

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^{\infty} e^{-st}\underbrace{\delta(t-t_0)}_{\lim\limits_{\tau\to 0} d_\tau(t-t_0)}\,dt$$



$$= \lim_{\tau\to 0}\int_0^{\infty} e^{-st} d_\tau(t-t_0)\,dt$$

$$= \lim_{\tau\to 0}\int_{t_0}^{t_0+\tau}\frac{1}{\tau}e^{-st}\,dt = \lim_{\tau\to 0}\frac{1}{\tau}\int_{t_0}^{t_0+\tau} e^{-st}\,dt$$

$$= \lim_{\tau\to 0}\frac{1}{\tau}\left(-\frac{1}{s}e^{-st}\Big|_{t_0}^{t_0+\tau}\right) = \lim_{t\to 0}\frac{1}{\tau}\cdot\frac{-1}{s}\left(e^{-s(t_0+\tau)}-e^{-st_0}\right)$$

$$= \frac{-1}{s}e^{-st_0}\underbrace{\lim_{\tau\to 0}\frac{e^{-s\tau}-1}{\tau}}_{\frac{d}{dt}e^{-st}\big|_{t=0}} \leftarrow \lim_{\tau\to 0}\frac{f(\tau)-f(0)}{\tau} = f'(0)$$

$$\frac{d}{dt}e^{-st}\Big|_{t=0} = \lim_{\tau\to 0}\frac{e^{-s\tau}-1}{\tau}$$

$$= -se^{-st}\Big|_{t=0}$$

$$= -s$$

$$= e^{-st_0}$$

eg1 $2y'' + y' + 2y = \delta(t-5) \quad y(0)=0 \quad y'(0)=0$

(This is mass-spring system with small damping, i.e. $\gamma^2 < 4mk$.)

$$2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-5)\}$$

$$2s^2\mathcal{L}\{y\} + s\mathcal{L}\{y\} + 2\mathcal{L}\{y\} = e^{-5s}$$

$$(2s^2+s+2)\mathcal{L}\{y\} = e^{-5s}$$

$$\mathcal{L}\{y\} = \frac{e^{-5s}}{2s^2+s+2} = \frac{e^{-5s}}{2\left[\left(s^2+\frac{1}{2}s+\frac{1}{16}\right)+1-\frac{1}{16}\right]}$$

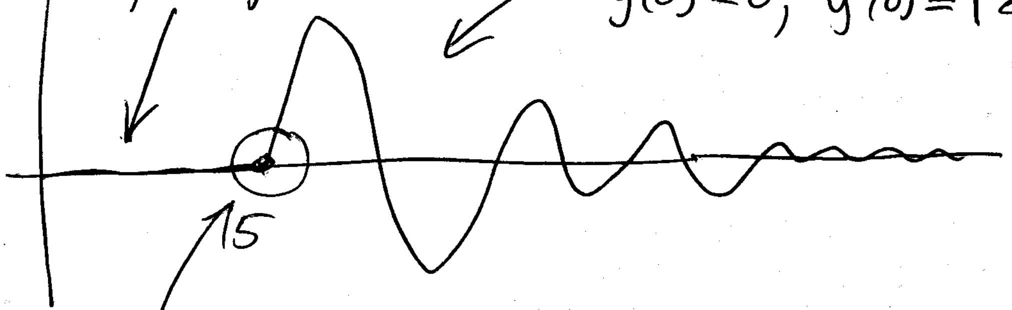
$$= \frac{1}{2}e^{-5s}\left[\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right]$$

$$\boxed{\mathcal{L}^{-1}\{[-]\} = \mathcal{L}^{-1}\left\{\frac{4}{\sqrt{15}} \cdot \frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right\} = \frac{4}{\sqrt{15}}e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right)}$$

$$y = \left[\frac{1}{\cancel{2}} \cdot \frac{\cancel{4}\,2}{\sqrt{15}}e^{-(t-5)/4}\sin\left(\frac{\sqrt{15}}{4}(t-5)\right)\right]u_5(t).$$

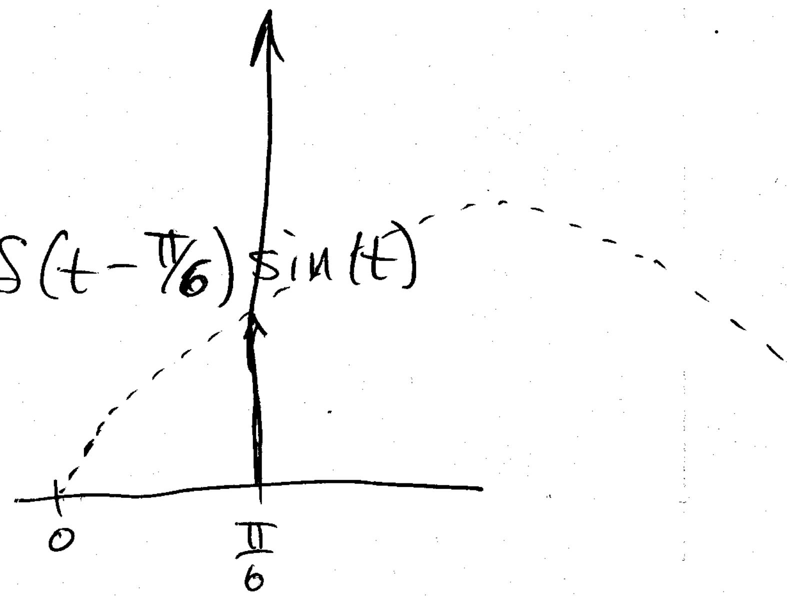
$2y'' + y' + 2y = 0$
$y(0) = y'(0) = 0$

$2y'' + y' + 2y = 0$
$y(0) = 0, \quad y'(0) = 1 \leftarrow$ velocity imparted by the impulse.



continuous but discont. first derivative (so a corner)

eg #(0) $2y'' + y' + 4y = \delta(t - \pi/6)\sin(t)$
$y(0) = y'(0) = 0$



$$= \delta(t - \tfrac{\pi}{6}) \cdot \sin(\tfrac{\pi}{6})$$
$$= \frac{1}{2}\delta(t - \tfrac{\pi}{6})$$

$$(2s^2 + s + 4)\mathcal{L}\{y\} = \frac{1}{2}e^{-\pi/6\, s}$$

$$\mathcal{L}\{y\} = \frac{1}{2}e^{-\pi/6\, s}\frac{1}{2\left[\left(s^2 + \frac{1}{2}s + \frac{1}{16}\right) + 2 - \frac{1}{16}\right]}$$

$$= \frac{1}{\cancel{4}} e^{-\pi/6\, s} \frac{\cancel{4}}{\sqrt{31}} \cdot \frac{\frac{\sqrt{31}}{4}}{(s + \frac{1}{4})^2 + \frac{31}{16}}$$

$$= \frac{1}{\sqrt{31}} e^{-\pi/6\, s}\left[\frac{\sqrt{31}/4}{(s + \frac{1}{4})^2 + \frac{31}{16}}\right]$$

$$\boxed{\mathcal{L}^{-1}\{[-]\} = e^{-t/4}\sin\left(\frac{\sqrt{31}}{4}t\right)}$$

$$y = \frac{1}{\sqrt{31}} e^{-(t-\pi/6)/4} \sin\left(\frac{\sqrt{31}}{4}(t-\pi/6)\right) u_{\frac{\pi}{6}}(t).$$

7.1 <u>Systems</u> of First Order Equations

<u>Idea:</u> $y^{(n)} = F(t, y, y', \ldots, y^{(n-1)})$

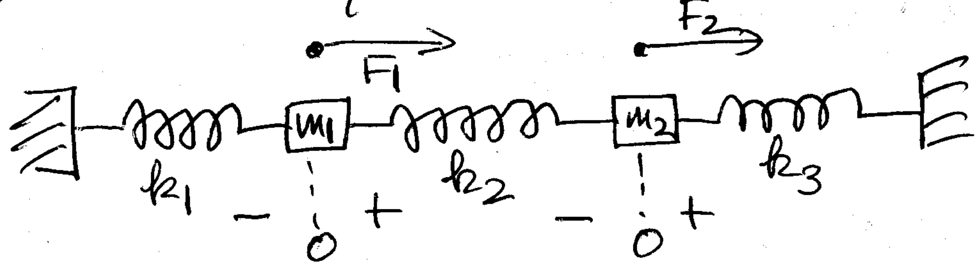
General n^{th} order ODE. Note that only one variable, y, describes the solution.

<u>e.g.</u> mass-spring: $mu'' + \gamma u' + ku = F$

$u(t)$ = position of mass at t.

What if 2 (or more) quantities are required to describe system?

<u>e.g.</u>



2-mass
3 springs.
<u>(no damping)</u>

$x_1(t)$ = position of m_1 at t.

$x_2(t)$ = " " m_2 at t.

(Say $x_1 > 0$) $m_1 x_1'' = -k_1 x_1 - k_2(x_1 - x_2) + F_1$

Similarly $m_2 x_2'' = -k_3 x_2 - k_2(x_2 - x_1) + F_2$.

$$\left.\begin{aligned} m_1 x_1'' &= -(k_1+k_2)x_1 + k_2 x_2 + F_1 \\ m_2 x_2'' &= +k_2 x_1 - (k_2+k_3)x_2 + F_2 \end{aligned}\right\} \circledast$$

<u>Any</u> D.E. can be written as a system of 1st-order DEs.

<u>e.g. 1</u> $\quad u'' + \frac{1}{8}u' + u = 0$

$$\begin{aligned} x_1 &= u \\ x_2 &= u' \end{aligned} \longrightarrow \begin{aligned} x_1' &= u' = x_2 \\ x_2' &= u'' = -\tfrac{1}{8}u' - u \\ &= -\tfrac{1}{8}x_2 - x_1 \end{aligned}$$

$$\longrightarrow \boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - \tfrac{1}{8}x_2. \end{aligned}}$$

<u>e.g.</u> Our example from earlier. $\circledast$

(use $y_1, y_2, \ldots$ as variables).

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_1' \\ y_3 &= x_2 \\ y_4 &= x_2' \end{aligned} \longrightarrow \begin{aligned} y_1' &= x_1' = y_2 \\ y_2' &= x_1'' = -\frac{k_1+k_2}{m_1}x_1 + \frac{k_2}{m_1}x_2 + F_1 \\ &= -\frac{k_1+k_2}{m_1}y_1 + \frac{k_2}{m_1}y_3 + F_1 \end{aligned}$$

$$y_3' = x_2' = y_4$$

$$y_4' = x_2'' = -\frac{k_2}{m_2} x_1 - \frac{k_2 + k_3}{m_2} x_2 + F_2$$

$$= -\frac{k_2}{m_2} y_1 - \frac{k_2 + k_3}{m_2} y_3 + F_2$$

System:

$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= \frac{-k_1 + k_2}{m_1} y_1 + \frac{k_2}{m_1} y_3 + F_1 \\ y_3' &= y_4 \\ y_4' &= -\frac{k_1 + k_3}{m_2} y_3 + \frac{k_2}{m_2} y_1 + F_2 . \end{aligned}}$$