Exam 1 - Tues 9/27

Coverage 1.1–1.3, 2.1–2.4, 2.6

No calculators, 3x5 card is permitted

Finish up 2.6 — Integrating Factors.

Exact equations: $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

Exact means $M(x,y)\,dx + N(x,y)\,dy$ is an exact differential, i.e. there is a function $\psi(x,y)$ such that $\frac{\partial \psi}{\partial x} = M(x,y)$, $\frac{\partial \psi}{\partial y} = N(x,y)$

Test for exactness: If $M_y = N_x$ then equation is exact.

If not exact, not much to do except to try one thing: Look for integrating factor $\mu(x,y)$.

Idea: $$M\,dx + N\,dy = 0$$

$$\mu M\,dx + \mu N\,dy = 0$$

Exact here means $(\mu M)_y = (\mu N)_x$

$$\mu M_y + \mu_y M = \mu N_x + \mu_x N$$

$$\mu(M_y - N_x) + \mu_y M - \mu_x N = 0$$

If can solve for μ then you have integrating factor.

In general this is harder than original problem so no help.

One hope: If $\mu(x,y) = \mu(x)$ (i.e. y does not appear in $\mu(x,y)$) then $\mu_y = 0$ and our equation becomes

$$\mu(M_y - N_x) - \mu_x N = 0 \quad \underline{\text{or}}$$

$$N \frac{d\mu}{dx} = \mu(M_y - N_x)$$

$$\frac{d\mu}{dx} = \mu\left(\frac{M_y - N_x}{N}\right)$$

If this is a function of x alone, we are in business.

$\underline{\text{e.g.}}$ $$\underbrace{(3xy + y^2)}_{M} + \underbrace{(x^2 + xy)}_{N} y' = 0$$

Exact? $M_y = 3x + 2y \quad N_x = 2x + y \quad \underline{\text{No}}$

$$\frac{M_y - N_x}{N} = \frac{3x + 2y - 2x - y}{x^2 + xy} = \frac{x + y}{x^2 + xy} = \frac{\cancel{x+y}}{x\cancel{(x+y)}}$$

$$= \frac{1}{x}$$ we can find μ.

Solve $\frac{d\mu}{dx} = \frac{1}{x}\mu \rightarrow \frac{d\mu}{\mu} = \frac{1}{x}\,dx \rightarrow \ln|\mu| = \ln|x| \cancel{+c}$

$c = 0$

take $\mu(x) = x$.

Now solve

$$x(3xy+y^2)+x(x^2+xy)y'=0$$

$$\underbrace{(3x^2y+xy^2)}_{M}+\underbrace{(x^3+x^2y)}_{N}y'=0$$

$$\begin{aligned} M_y &= 3x^2+2xy \\ N_x &= 3x^2+2xy \end{aligned} \Big\rangle \text{ I knew it all along } \smile$$

Find ψ $\quad \dfrac{\partial\psi}{\partial x}=3x^2y+xy^2$

$$\psi=x^3y+\tfrac{1}{2}x^2y^2+g(y)$$

$$\psi_y=\cancel{x^3}+\cancel{x^2y}+g'(y)=\cancel{x^3}+\cancel{x^2y}$$

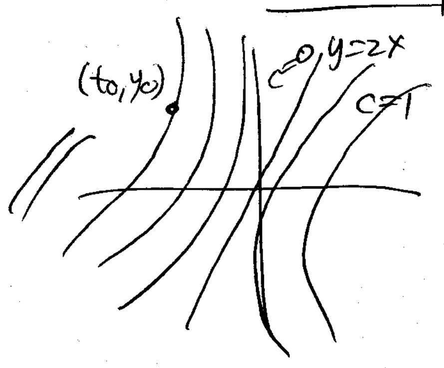
$$g'(y)=0 \qquad g(y)=\text{const (take } g(y)=0).$$

$$\therefore\ \psi(x,y)=x^3y+\tfrac{1}{2}x^2y^2$$

$$\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}\frac{dy}{dx}=0 \rightarrow \frac{d}{dx}\big(\psi(x,y(x))\big)=0$$

$$\rightarrow \psi(x,y(x))=c$$

Solution: $x^3y+\frac{1}{2}x^2y^2=c$



e.g. #28 (p.101)

$$\underbrace{y}_{M}\,dx + \underbrace{(2xy - e^{-2y})}_{N}\,dy = 0$$

$$M_y = 1 \quad N_x = 2y$$

Look at: $\dfrac{M_y - N_x}{N} = \dfrac{1-2y}{2xy - e^{-2y}}$?

This does not work!

$$\mu(M_y - N_x) + \mu_y M - \overset{0}{\cancel{\mu_x}} N = 0$$

Assume $\mu(x,y) = \mu(y)$

$$\mu_y = \mu\left(\frac{N_x - M_y}{M}\right) \qquad \frac{d\mu}{dy} = \mu\left(\frac{N_x - M_y}{M}\right)$$

In our case

$$\frac{N_x - M_y}{M} = \frac{2y-1}{y} = 2 - \frac{1}{y}$$

Solve: $\dfrac{d\mu}{dy} = 2\mu - \dfrac{1}{y}(\mu) = \mu\left(2 - \dfrac{1}{y}\right)$

$$\frac{d\mu}{\mu} = \left(2 - \frac{1}{y}\right)dy \rightarrow \ln|\mu| = 2y - \ln|y|$$

$$\mu = e^{2y - \ln(y)} = e^{2y}e^{-\ln y} = \underline{\underline{\frac{1}{y}e^{2y}}}$$

New problem is:

$$\frac{1}{y} e^{2y}\left(y\,dx + (2xy - e^{-2y})\,dy\right) = 0$$

$$\overset{M}{e^{2y}}\,dx + \overset{N}{\left(2xe^{2y} - \frac{1}{y}\right)}\,dy = 0$$

Should be exact:

$M_y = 2e^{2y}$ $\quad N_x = 2e^{2y}$ ✓

etc----

3.1 Homogeneous Equations with constant coefficients.

Now considering second order equations of the form
$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$
Generally hard to solve, so we restrict our attention to:

(i) linear equations, i.e.
$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$
Can also be written
$$y'' + p(t)y' + q(t)y = g(t).$$
by dividing by $P(t)$, AND

(ii) homogeneous equations, i.e. $G(t) = 0$
$$P(t)y'' + Q(t)y' + R(t)y = 0 \quad \text{or}$$
$$y'' + p(t)y' + q(t)y = 0, \text{ AND}$$

(iii) constant coefficients, i.e. P, Q, R are constant so we are solving
$$ay'' + by' + cy = 0 \leftarrow \text{plenty to do here.}$$

What about initial values?

Before: $\frac{dy}{dt} = f(t,y)$ $\quad y(t_0) = y_0$

Now: $\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$ $\quad \left.\begin{array}{l} y(t_0) = y_0 \\ y'(t_0) = y_0' \end{array}\right\}$ Why 2?

e.g. solve $\underline{y'' = 0}$

$y' = c_1$

$y = c_1 t + c_2$ ← 2 arbitrary constants.

So eg. if $\underline{y(0) = 2}$ $\underline{y'(0) = 1}$ we get.

$2 = c_1 \cdot 0 + c_2 = c_2$ ✓

$y' = c_1$ $\quad 1 = c_1$ ✓

Solution is: $\boxed{y = t + 2}$

e.g. $y'' = y \quad y(0) = 2 \quad y'(0) = -1$

or $y'' - y = 0$

Take a stab at a solution:

$y = e^t \quad y' = e^t \quad y'' = e^t$

$y = ce^t \quad y' = ce^t \quad y'' = ce^t$

Any others? We expect so since only one arbitrary constant.

How about $y = ce^{-t}$?

$y = ce^{-t} \quad y' = -ce^{-t} \quad y'' = (-1)^2 ce^{-t} = ce^{-t}$

Now have 2 solutions: $y_1(t) = c_1 e^t \quad y_2(t) = c_2 e^{-t}$

Can I combine them into one solution?

Try $y(t) = c_1 e^t + c_2 e^{-t}$. Does this work?

$y' = c_1 e^t - c_2 e^{-t} \quad y'' = c_1 e^t + c_2 e^{-t}$ OK

General fact about homogeneous linear equations: If y_1, y_2 are both solutions then so is $c_1 y_1 + c_2 y_2$ for any constants c_1, c_2.

Do we need to keep looking? Let's stop because we have 2 arb. constants already.

Lets solve the IVP:

General solution $y(t) = c_1 e^t + c_2 e^{-t}$

$y(0) = 2 \quad y'(0) = -1$. Find c_1, c_2.

$2 = c_1 + c_2 \qquad y'(t) = c_1 e^t - c_2 e^{-t}$

$-1 = c_1 - c_2$

Solve system
$$\begin{array}{r} c_1 + c_2 = 2 \\ c_1 - c_2 = -1 \\ \hline 2c_1 = 1 \end{array}$$

$c_1 = \frac{1}{2} \quad c_2 = 2 - \frac{1}{2} = \frac{3}{2}$

Solution is: $y(t) = \frac{1}{2} e^t + \frac{3}{2} e^{-t}$

Suggests a general procedure:

Look for solutions of the form $y = e^{rt}$

In our case: $y'' - y = 0 \qquad y' = re^{rt}$

$y'' = r^2 e^{rt}$

$r^2 e^{rt} - e^{rt} = 0$

$(r^2 - 1) e^{rt} = 0$

$r^2 - 1 = 0 \longrightarrow r = 1$

$r = -1$

I conclude $y = e^t$ and $y = e^{-t}$ work.