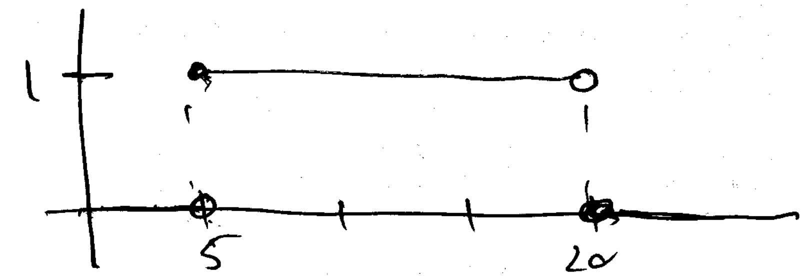
Quiz 12 - Section 6.2

Discontinuous forcing functions.

e.g. $2y'' + y' + 2y = \underbrace{u_5(t) - u_{20}(t)}_{\text{forcing function}}$

$y(0) = 0, \quad y'(0) = 0.$



$$\mathcal{L}\{y\} = \frac{1}{2}\left(e^{-5s} - e^{-20s}\right)\left(\frac{1}{s} - \frac{s+\frac{1}{4}}{(s+\frac{1}{4})^2 + \frac{15}{16}} - \frac{1}{4}\frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}}\right)$$

$$= \frac{1}{2}\left(e^{-5s} - e^{-20s}\right)\left[\frac{1}{s} - \frac{s+\frac{1}{4}}{(s+\frac{1}{4})^2 + \frac{15}{16}} - \frac{1}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2 + \frac{15}{16}}\right]$$

$$\mathcal{L}^{-1}\{[-]\} = 1 - e^{-t/4}\cos\left(\frac{\sqrt{15}}{4}t\right) - \frac{1}{\sqrt{15}}e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

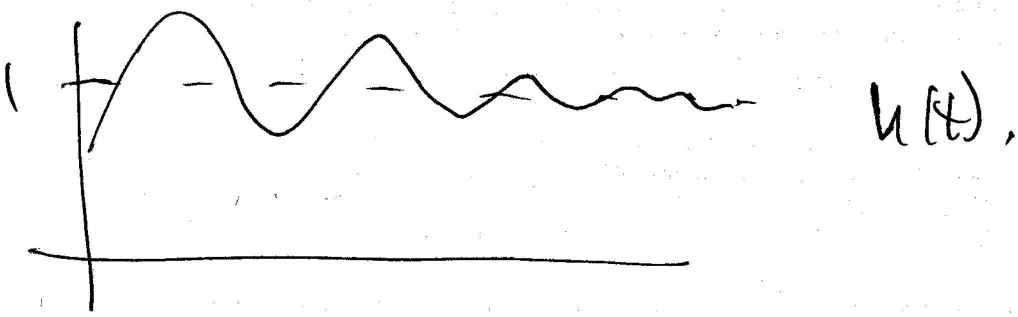
$$y = \frac{1}{2}u_5(t)\left(1 - e^{-(t-5)/4}\cos\frac{\sqrt{15}}{4}(t-5) - \frac{1}{\sqrt{15}}e^{-(t-5)/4}\sin\left(\frac{\sqrt{15}}{4}(t-5)\right)\right)$$

$$- \frac{1}{2}u_{20}(t)\left[1 - e^{-(t-20)/4}\cos\frac{\sqrt{15}}{4}(t-20) - \frac{1}{\sqrt{15}}e^{-(t-20)/4}\sin\left(\frac{\sqrt{15}}{4}(t-20)\right)\right]$$

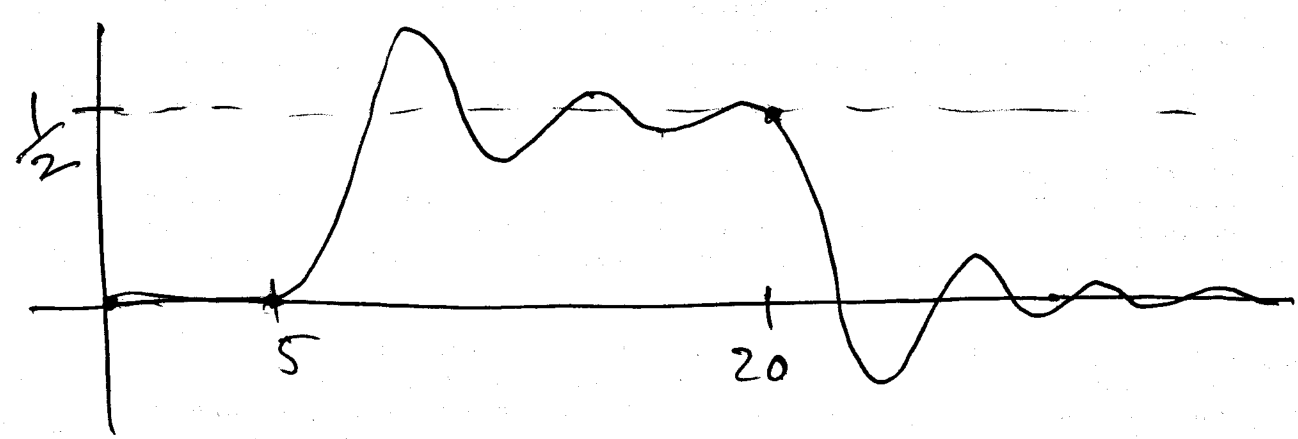
Qualitatively:

$$h(t) = 1 - e^{-t/4} \cos \frac{\sqrt{15}}{4} t - \frac{1}{\sqrt{15}} e^{-t/4} \sin \frac{\sqrt{15}}{4} t.$$

$h(t)$ = damped oscillation about a constant



$$y = \left(u_5(t) h(t-5) - u_{20}(t) h(t-20) \right) \cdot \frac{1}{2} .$$



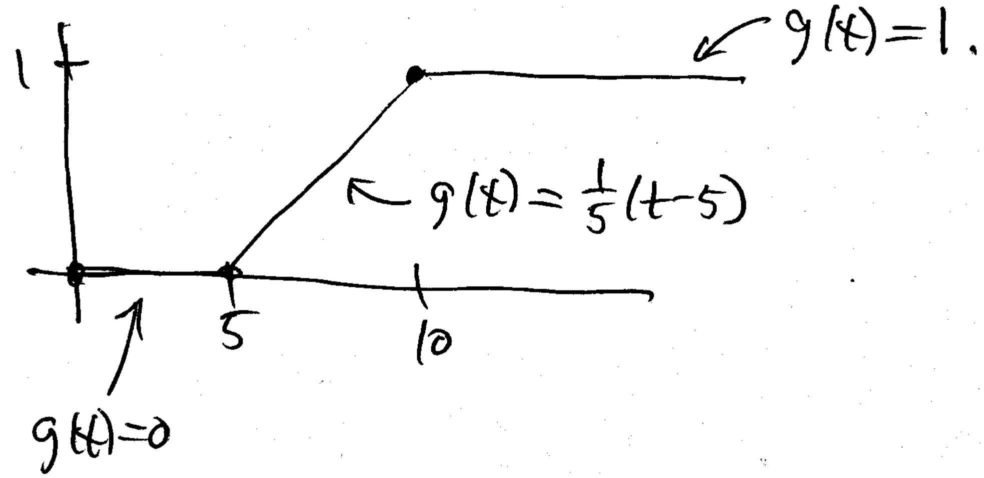
$0 \le t < 5$ no movement, $y \equiv 0$.

$5 \le t < 20$ shift of $h(t)$ (ie $h(t-5)$).

$20 \le t$ $y(t) = h(t-5) - h(t-20)$
damped oscillation about zero.

e.g. 3 $\quad y'' + 4y = g(t) \qquad y(0) = y'(0) = 0$

$g(t)$:



$$g(t) = 0 + \frac{1}{5}(t-5)\,u_5(t) + \left(1 - \frac{1}{5}(t-5)\right)u_{10}(t)$$

$$= \frac{1}{5}(t-5)\,u_5(t) + \frac{1}{5}(10-t)\,u_{10}(t)$$

$$= \frac{1}{5}(t-5)\,u_5(t) - \frac{1}{5}(t-10)\,u_{10}(t).$$

Laplace transform of both sides of ODE:

$$s^2\mathcal{L}\{y\} + 4\mathcal{L}\{y\} = \frac{1}{5}e^{-5s}\mathcal{L}\{t\} - \frac{1}{5}e^{-10s}\mathcal{L}\{t\}$$

$$\mathcal{L}\{y\}(s^2+4) = \frac{1}{5}\left(e^{-5s} - e^{-10s}\right)\cdot\frac{1}{s^2}.$$

$$\mathcal{L}\{y\} = \frac{1}{5}\left(e^{-5s} - e^{-10s}\right)\frac{1}{s^2(s^2+4)}$$

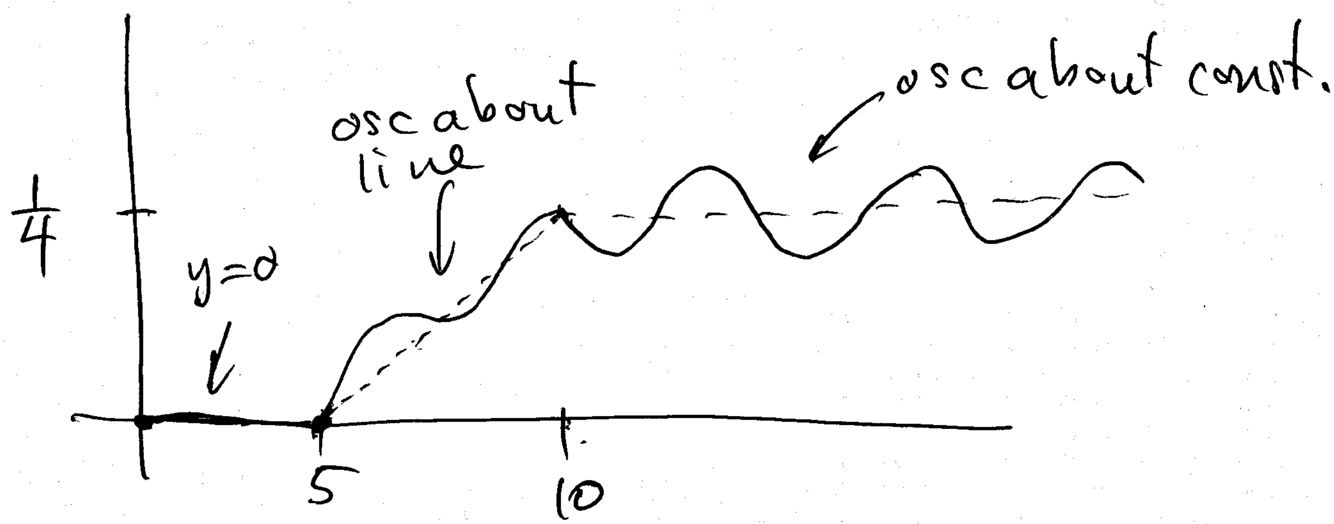
$$\frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4} = \cdots = \frac{1}{4}\cdot\frac{1}{s^2} - \frac{1}{4}\cdot\frac{1}{s^2+4}$$

$$\frac{1}{(s-a)^n(s-b)^m} = \frac{A}{(s-a)} + \frac{B}{(s-a)^2} + \cdots + \frac{C}{(s-a)^n} + \frac{D}{(s-b)} + \frac{E}{(s-b)^2} + \cdots + \frac{F}{(s-b)^m}$$

$$\mathcal{L}\{y\} = \frac{1}{20}\left(e^{-5s} - e^{-10s}\right)\left[\frac{1}{s^2} - \frac{1}{2}\frac{2}{s^2+4}\right]$$

$$\mathcal{L}^{-1}\{[-]\} = t - \frac{1}{2}\sin(2t)$$

$$y = \frac{1}{20}\Big(\left((t-5) - \tfrac{1}{2}\sin(2(t-5))\right)u_5(t)$$
$$- \left((t-10) - \tfrac{1}{2}\sin(2(t-10))\right)u_{10}(t)\Big)$$



$0 \le t < 5$	$y = 0$
$5 \le t < 10$	oscillation about $\frac{1}{20}(t-5)$
$10 \le t$	oscillation about $\frac{1}{20}(t-5) - \frac{1}{20}(t-10) = \frac{1}{4}$.

eg. #6) $y'' + 3y' + 2y = u_2(t) \qquad y(0) = 0 \quad y'(0) = 1.$

(overdamped)

$$s^2\mathcal{L}\{y\} - s\cancelto{0}{y(0)} - \cancelto{1}{y'(0)} + 3s\mathcal{L}\{y\} - 3\cancelto{0}{y(0)} + 2\mathcal{L}\{y\} = \mathcal{L}\{u_2\}$$

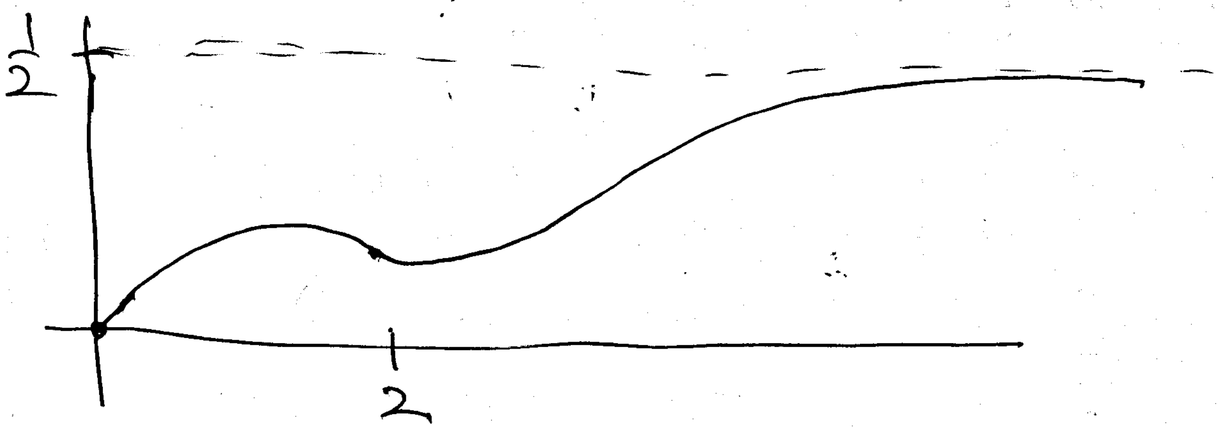
$$(s^2+3s+2)\mathcal{L}\{y\} = 1 + \frac{1}{s}e^{-2s}$$

$$\mathcal{L}\{y\} = \frac{1}{(s+1)(s+2)} + e^{-2s}\frac{1}{s(s+1)(s+2)}$$

$$= \frac{A}{s+1} + \frac{B}{s+2} + e^{-2s}\left(\frac{C}{s} + \frac{D}{s+1} + \frac{E}{s+2}\right)$$

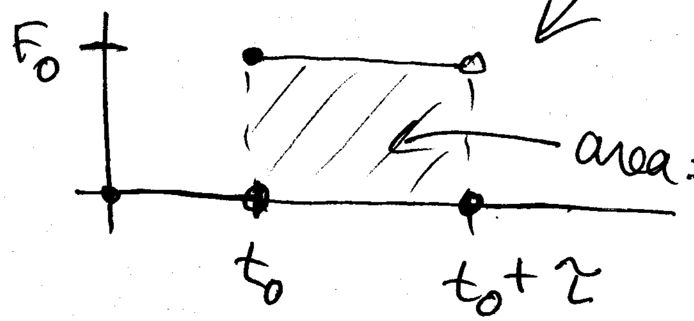
$$= \left(\frac{1}{s+1} - \frac{1}{s+2}\right) + e^{-2s}\left(\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\,\frac{1}{s+2}\right)$$

$$y = \left(e^{-t} - e^{-2t}\right) + \left(\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right)u_2(t)$$



6.5 Impulse Functions

$$mu'' + \gamma u' + ku = F$$



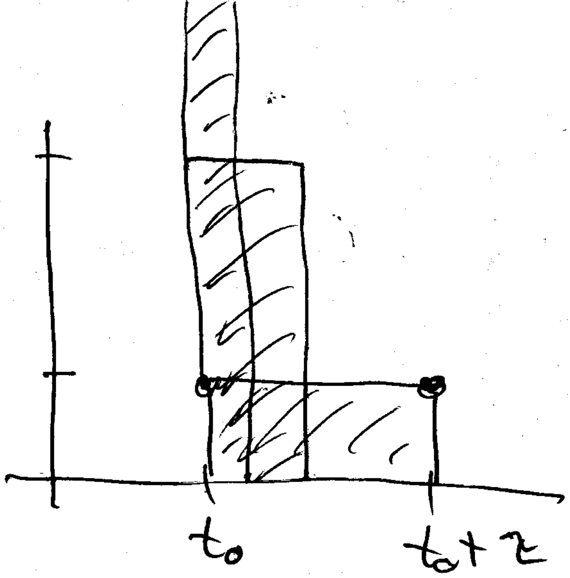
area $= F_0 \cdot \tau$ for example.

How do we interpret area under the curve?

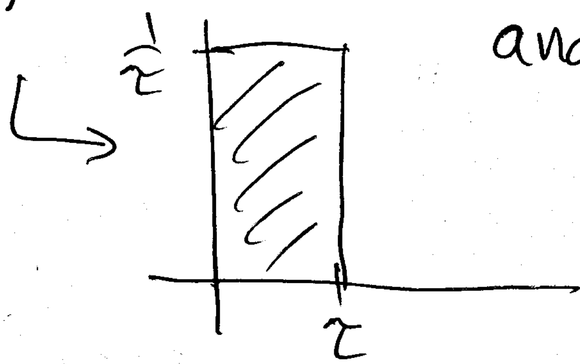
1. Newton's second law: $F = ma$
2. One ~~Net~~ Newton of force accelerates a 1 kg mass at 1 m/s^2 (so $N = \frac{kg\text{-}m}{s^2}$).
3. If F_0 newtons act on a 1 kg mass for τ seconds, the change in velocity is $(F_0 \cdot \tau)\ \frac{m}{sec}$ which is area under curve.

This is called the impulse of the force.

How would we achieve same change in velocity over shorter time interval?



consider the unit impulse over $[0, \tau]$, called $d_\tau(t)$ and we let $\tau \to 0^+$.



Define $\delta(t) = \lim_{\tau \to 0^+} d_\tau(t)$. $\delta(t)$ satisfies

1) $\delta(t) = 0$ if $t \neq 0$, 2) $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

No such function exists but we use this to model impulses of very short duration.