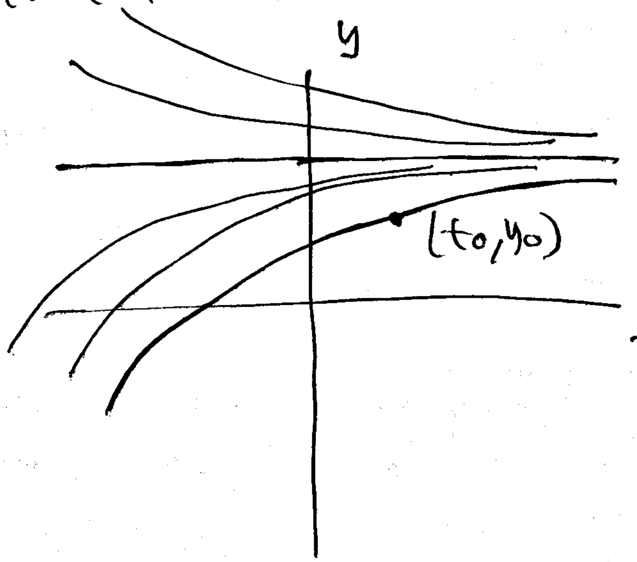
Quiz 3 on Thursday will be on 2.1 and 2.2.

2.4 Linear v.s Nonlinear ODEs (Existence and Uniqueness)

Looking at $\frac{dy}{dt} = f(t,y)$ $\quad \underline{y(t_0)=y_0}$ (IVP)

1. Can we guarantee that a solution exists?
2. Can we guarantee that there is only one solution?

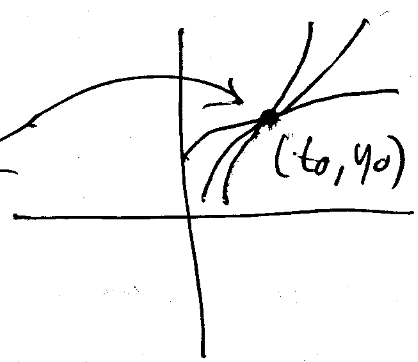
Remember this is an IVP.



In all examples we have seen there is only <u>one</u> solution through each point (t_0,y_0).

Can this happen:

know that the slope here must be $f(t_0,y_0)$



2 solutions thru same point?

3. Can we always find the interval of t where the solution (or solutions) exists?

(We have seen examples where the solution exists for some t but not all t.)

e.g. $ty' + 2y = 4t^2 \quad y(1) = 2$

Already solved this one:

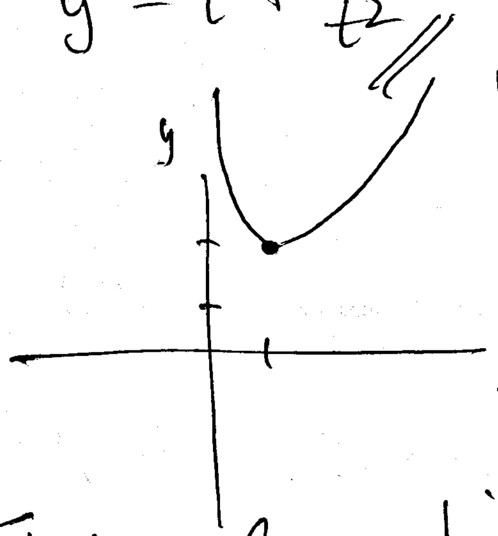
$$y' + \frac{2}{t} y = 4t \qquad \frac{d}{dt}(t^2 y) = 4t \cdot t^2 = 4t^3$$

$$\mu = e^{\int \frac{2}{t} dt} = e^{2 \ln t} = t^2 \qquad t^2 y = t^4 + c$$

$$y = t^2 + \frac{c}{t^2}.$$

$$y(1) = 2 \rightarrow 2 = 1 + c \rightarrow c = 1.$$

$$y = t^2 + \frac{1}{t^2}$$



solution only exists for $t > 0$.

* [Interval on which solutions exist depends on the equation and on the initial condition]

Return to the uniqueness question.

<u>e.g.</u> $\frac{dy}{dt} = y^{1/3}$ $y(0)=0$. Without doing any work we see that $y(t)=0$ is a solution.

non-linear separable.

$\frac{dy}{y^{1/3}} = dt$

$\frac{3}{2}y^{2/3} = t + c$

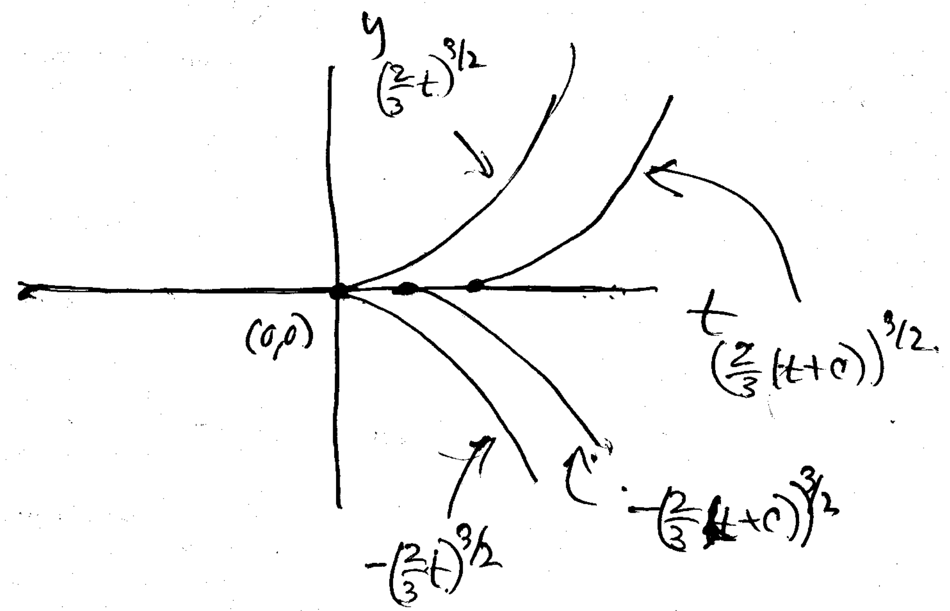
$y(0)=0$

$c=0$.

$\frac{3}{2}y^{2/3} = t$

$y = \left(\frac{2}{3}t\right)^{3/2}$

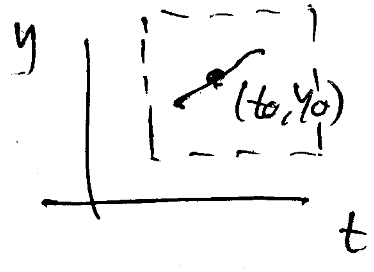
valid for $t \geq 0$ only.



So what is the bottom line?

<u>Theorem 2.4.2</u> $\frac{dy}{dt} = f(t,y)$ $y(t_0)=y_0$

If f, $\frac{\partial f}{\partial y}$ continuous in some rectangle containing (t_0,y_0) $\alpha<t<\beta$, $\gamma<y<\delta$ then there is unique solution on <u>some</u> interval within (α,β).

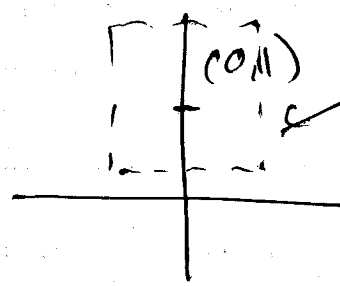


Think about $y' = y^{1/3}$ $\quad y(0)=0.$

$f(t,y) = y^{1/3}$, $\quad \frac{\partial f}{\partial y} = \frac{1}{3}y^{-2/3}$ ← not continuous in any interval containing $y=0$.

Suppose instead we take $\underline{y(0)=1}$

Then we are $\underline{\text{OK}}$.



f, $\frac{\partial f}{\partial y}$ cont here.

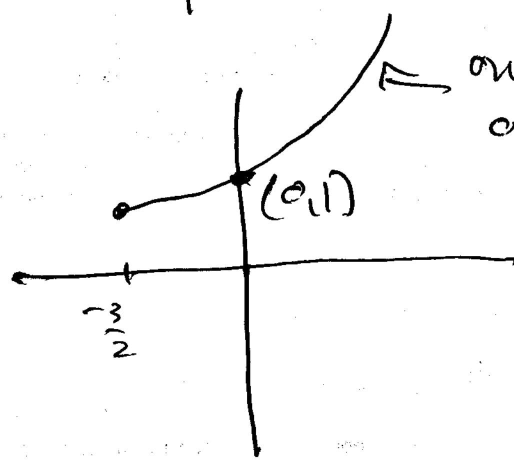
$$\frac{3}{2}y^{2/3} = t + c \qquad y(0)=1$$

$$\frac{3}{2} = c$$

$$\frac{3}{2}y^{2/3} = t + \frac{3}{2}$$

$$y = \left(\frac{2}{3}t + 1\right)^{3/2}$$

valid for $t > -\frac{3}{2}$



← one solution only

For linear equations we have:

$$\frac{dy}{dt} + p(t)y = g(t) \qquad y(t_0) = y_0$$

$$\frac{dy}{dt} = \underbrace{-p(t)y + g(t)}_{f(t,y)} \qquad \frac{\partial f}{\partial y} = -p(t)$$

so if $p(t)$ is continuous in an interval around t_0, we have a unique solution. Also $g(t)$ needs to be continuous.

e.g. #2

$$t(t-4)y' + y = 0 \rightarrow y' + \frac{1}{t(t-4)}y = 0$$

$y(2) = 1$. $p(t)$ is continuous only on $(-\infty, 0) \cup (0,4) \cup (4,\infty)$.

Since $t_0 = 2 \in (0,4)$ I can guarantee a solution on $(0,4)$.

eg if $y(2) = 0$ then I get a solution for all t.

$y(2) = 0$ then the solution is $y(t) = 0$.

#5. $(4-t^2)y' + 2ty = 3t^2 \quad y(1) = -3$

$$y' + \frac{2t}{4-t^2}y = \frac{3t^2}{4-t^2}$$

Unique solutions on the intervals.

$(-\infty,-2), (-2,2), (2,\infty)$.

Since $t_0 = 1$, I can guarantee a unique solution on $(-2,2)$.

#14. $y' = 2ty^2 \quad y(0) = y_0$.

nonlinear

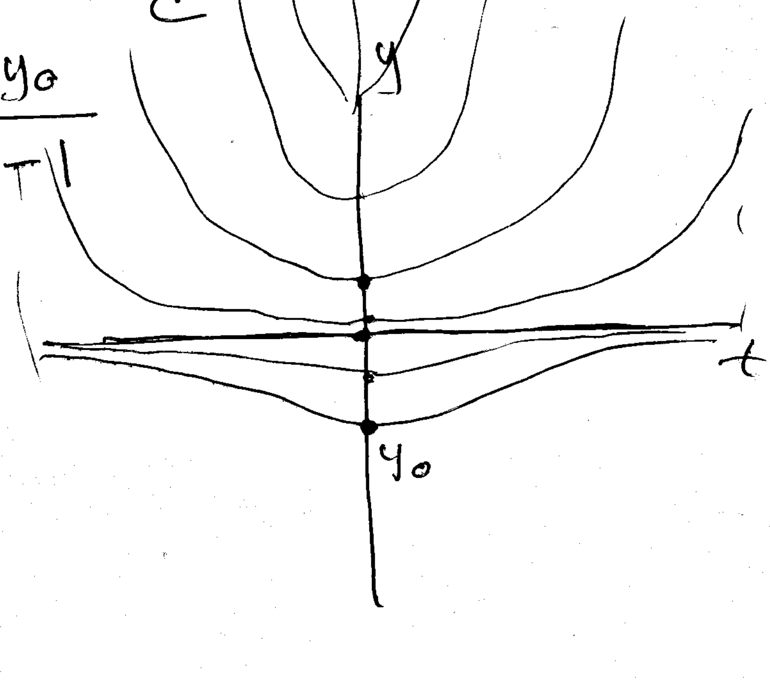
$$\frac{dy}{y^2} = 2t\,dt$$

$$-\frac{1}{y} = t^2 + c$$

$$y = \frac{-1}{t^2+c} \qquad y(0) = y_0$$

$$y_0 = \frac{-1}{c} \rightarrow c = -\frac{1}{y_0}$$

$$y = \frac{-1}{t^2 - \frac{1}{y_0}} = \frac{-y_0}{y_0 t^2 - 1}$$



If $y_0 \le 0$ solution exists for all t.

If $y_0 > 0$ then solution exists only for $|t| < \frac{1}{\sqrt{y_0}}$

2.6 Exact Equations & Integrating Factors.

Exact differential

vector field: $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$

Q: Is $\vec{F}$ a gradient field?

that is, is $\vec{F} = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$

Is there a function $f(x,y)$ such that

$\frac{\partial f}{\partial x} = M(x,y)$ and $\frac{\partial f}{\partial y} = N(x,y)$?

If so we say $M(x,y)dx + N(x,y)dy$ is an exact differential.

Exact equation has the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

In general these are not linear and not separable.

These can be solved if $Mdx + Ndy$ is an exact differential.

Suppose that $M = \frac{\partial \psi}{\partial x}$, $N = \frac{\partial \psi}{\partial y}$. Then eqn. becomes

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{d}{dx}\left(\psi(x, y(x))\right) = 0$$

$$\psi(x,y) = c.$$ solution (implicit solution)

eg $\underbrace{(2x+y^2)}_{M}+\underbrace{(2xy)}_{N}y'=0$

Look for $\psi(x,y)$ s.t. $M=\dfrac{\partial\psi}{\partial x}$, $N=\dfrac{\partial\psi}{\partial y}$

Solve $\dfrac{\partial\psi}{\partial x}=2x+y^2$ $\dfrac{\partial\psi}{\partial y}=2xy$

$\psi=x^2+xy^2+g(y)$

$\xrightarrow{\frac{\partial\psi}{\partial y}}$ $2xy+g'(y)=2xy$

$g'(y)=0$ $g=\text{const.}$

I can take $g=0$ since I only need a solution not all solutions.

Solution: $\psi(x,y)=x^2+xy^2$

solution to ODE is $\underbrace{x^2+xy^2=c}_{\text{implicit form}}$.

or $y^2=\dfrac{c-x^2}{x}$ $\underbrace{y=\pm\left(\dfrac{c-x^2}{x}\right)^{1/2}}_{\text{explicit form.}}$

e.g. $\underbrace{(y\cos x + 2xe^y)}_{M} + \underbrace{(\sin x + x^2e^y - 1)}_{N}y' = 0$

Can I tell without actually solving, that this is exact? Yes.

$M = \frac{\partial \psi}{\partial x}$ $N = \frac{\partial \psi}{\partial y}$?

$M_y = \frac{\partial^2 \psi}{\partial y \partial x}$ $N_x = \frac{\partial^2 \psi}{\partial x \partial y}$

Thm: $M\,dx + N\,dy$ is exact differential if $M_y = N_x$.

$M_y = \cos x + 2xe^y$ $\therefore$ exact.

$N_x = \cos x + 2xe^y$

$\frac{\partial \psi}{\partial x} = y\cos x + 2xe^y$

$\psi = y\sin x + x^2e^y + g(y)$

$\frac{\partial \psi}{\partial y} = \cancel{\sin x} + \cancel{x^2e^y} + g'(y) = \cancel{\sin x} + \cancel{x^2e^y} - 1$

$g'(y) = -1$ so $g(y) = -y$

$\therefore \psi = y\sin x + x^2e^y - y$ $\therefore$ solution is: $y\sin x + x^2e^y - y = c$.