## 4.1 $n^{th}$-order linear ODE's.

Reminders:

Linear: $y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + p_2(t)y^{(n-2)}(t) + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t).$

$$L[y] = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1}y' + p_n y.$$

Homogeneous: $L[y] = 0$.

Initial conditions:

$$y(t_0) = y_0,\ y'(t_0) = y_0',\ y''(t_0) = y_0'',\ \ldots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}.$$

($n$ conditions all-together).

e.g. $y''' + t^2 y'' + e^t y' + 6y = 0$

$3^{rd}$ order, linear, homogeneous

$$y(0) = 1,\ y'(0) = 2,\ y''(0) = -1.$$

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Basic solution strategy for $L[y] = g$.

1) Solve homogeneous equation $L[y] = 0$.
   - Find $n$ solutions $y_1, y_2, \ldots, y_n$.
   - Verify that $\{y_1, y_2, \ldots, y_n\}$ forms a fundamental set (more later)
   - General solution

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t).$$

- coefficients $c_1, \dots, c_n$ may be determined by initial conditions.

2) Find a particular solution to $L[y] = g$, call it $Y$.

- General solution is

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t) + Y(t).$$

- Every solution looks like this.

Recall: If $Y_1$ is another solution then

$$L[Y_1 - Y] = L[Y_1] - L[Y] = g - g = 0. \text{ So}$$

$$Y_1 - Y = c_1 y_1(t) + \cdots + c_n y_n(t)$$

$$Y_1 = c_1 y_1(t) + \cdots + c_n y_n(t) + Y(t).$$

- coefficients are determined by initial conditions.

3.) Theorem 4.1.1

eg #2) $t y''' + (\sin(t)) y'' + 3y = \cos(t)$

$$y''' + \frac{\sin t}{t} y'' + \frac{3}{t} y = \frac{\cos(t)}{t}.$$

Solutions exist on $(-\infty, 0) \cup (0, \infty)$.

What about Wronskians, etc?

e.g. $y''' - 5y'' + 7y' - 3y = 0$

Solve: $r^3 - 5r^2 + 7r - 3 = 0$

$r = 1$ is a solution: Factoring we get

$$\begin{array}{r|l} & r^2 - 4r + 3 \\ r-1 & r^3 - 5r^2 + 7r - 3 \\ & -(r^3 - r^2) \\ \hline & -4r^2 + 7r \\ & -(-4r^2 + 4r) \\ \hline & 3r - 3 \end{array}$$

$(r-1)(r^2-4r+3) = 0$

$(r-1)(r-1)(r-3) = 0$

$(r-1)^2(r-3) = 0$

$r = 1,\ r = 3$

$y_1 = e^t,\ y_2 = te^t,\ y_3 = e^{3t}$

$y = c_1e^t + c_2te^t + c_3e^{3t}$

Initial conditions: $y(0) = 1,\ y'(0) = -1,\ y''(0) = 0.$

$y' = c_1e^t + c_2(te^t + e^t) + 3c_3e^{3t}$

$y'' = c_1e^t + c_2(te^t + 2e^t) + 9c_3e^{3t}$

$$2\left[\begin{array}{l} 1 = c_1 + c_3 \\ (-1 = c_1 + c_2 + 3c_3) \\ 0 = c_1 + 2c_2 + 9c_3 \end{array}\right] \longrightarrow$$

$1 = c_1 + c_3$

$2 = -c_1 + 3c_3$

$3 = 4c_3 \Rightarrow c_3 = \frac{3}{4}$

$c_1 = \frac{1}{4},\ c_2 = -\frac{7}{2}$

$y = \frac{1}{4}e^t - \frac{7}{2}te^t + \frac{3}{4}e^{3t}$ //

Writing as a matrix, we get.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 1 & 2 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Or in general:

$$\begin{bmatrix} y_1(0) & y_2(0) & y_3(0) \\ y_1'(0) & y_2'(0) & y_3'(0) \\ y_1''(0) & y_2''(0) & y_3''(0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \\ y_0'' \end{bmatrix}$$

Unique solution only if $\det\begin{bmatrix} \quad \end{bmatrix} \neq 0$

<u>Def</u>: Given $y_1, y_2, \dots, y_n$, the <u>Wronskian</u> is

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

In our example:

$$W(e^t, te^t, e^{3t}) = \begin{vmatrix} e^t & te^t & e^{3t} \\ e^t & te^t+e^t & 3e^{3t} \\ e^t & te^t+2e^t & 9e^{3t} \end{vmatrix}$$

$$= e^t \begin{vmatrix} te^t+e^t & 3e^{3t} \\ te^t+2e^t & 9e^{3t} \end{vmatrix} - te^t \begin{vmatrix} e^t & 3e^{3t} \\ e^t & 9e^{3t} \end{vmatrix}$$

$$+ e^{3t} \begin{vmatrix} e^t & te^t+e^t \\ e^t & te^t+2e^t \end{vmatrix}$$

$$= e^t\left(9te^{4t} + 9e^{4t} - 3te^{4t} - 6e^{4t}\right)$$

$$- te^t\left(9e^{4t} - 3e^{4t}\right) + e^{3t}\left(\cancel{te^{2t}} + 2e^{2t} - \cancel{te^{2t}} - e^{2t}\right)$$

$$= e^{5t}\left(\cancel{6t}+3\right) + \cancel{e^{5t}\left(-9t+3t\right)} + e^{5t}(1)$$

$$= 4e^{5t} \neq 0 \text{ for all } t.$$

Is there an easier way? <u>YES</u>.

1) Abel's Theorem still holds, i.e. $W$ satisfies

$$W' = -p_1 W \longrightarrow W = c\, e^{-\int p_1(t)\,dt}$$

In our example, $p_1 = -5$ so

$$W' = 5W \longrightarrow W = c\, e^{5t}$$

2) Linear Independence.

Functions $f_1, f_2, \dots, f_n$ are <u>linearly independent</u> on an interval $I$ if for all $t \in I$,

$$k_1 f_1(t) + k_2 f_2(t) + \cdots + k_n f_n(t) = 0$$

implies $k_1 = k_2 = \cdots = k_n = 0$. Otherwise, they are <u>linearly dependent</u>.

<u>e.g.</u> $1, t, 3+2t$ are linearly <u>dependent</u>.

$$k_1 \cdot 1 + k_2 \cdot t + k_3(3+2t) = 0 \leftarrow$$ this means must be 0 <u>for all $t$</u>.

$$(k_1 + 3k_3) + t(k_2 + 2k_3) = 0$$

$$k_1 + 3k_3 = 0$$
$$k_2 + 2k_3 = 0$$

2 equations in 3 unknowns.

By inspection:

$$\boxed{\begin{aligned} k_1 &= -3 \\ k_2 &= -2 \\ k_3 &= 1 \end{aligned}}$$

e.g. $1, t, t^2$ are linearly independent.

$$k_1 \cdot 1 + k_2 \cdot t + k_3 \cdot t^2 = 0$$

$$\boxed{k_1 + t k_2 + t^2 k_3 = 0}$$

Try some strategic values of $t$.

$t=0$: $\boxed{k_1 = 0}$

We have now: $t k_2 + t^2 k_3 = 0$

Derivative of both sides: $k_2 + 2t k_3 = 0$

$t=0$: $\boxed{k_2 = 0}$ $\longrightarrow$ $2t k_3 = 0$ $\quad$ $2 k_3 = 0$

$t=1$: $\boxed{k_3 = 0}$

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Theorem 4.1.3

$y_1, y_2, \cdots, y_n$ linearly independent on $I$ $\iff$ $W(y_1, \ldots, y_n) \neq 0$ on $I$.

~~Ex~~ Known example.

Show $e^t, te^t, e^{3t}$ linearly independent

$k_1 e^t + k_2 te^t + k_3 e^{3t} = 0$ Divide by $e^t$

$k_1 + k_2 t + k_3 e^{2t} = 0 \longrightarrow \boxed{k_1 = 0}$

Derivative: $k_2 + 2k_3 e^{2t} = 0 \rightarrow \boxed{k_2 = 0}$

2nd derivative: $4k_3 e^{2t} = 0$

$t = 0 : 4k_3 = 0 \longrightarrow \boxed{k_3 = 0}$

Conclusion $\{e^t, te^t, e^{3t}\}$ is a fundamental set of solutions.

e.g. #16) $x^3 y''' + x^2 y'' - 2xy' + 2y = 0$

$x, x^2, \frac{1}{x}$.

Linearly independent? Say for $x > 0$.

$k_1 x + k_2 x^2 + \frac{k_3}{x} = 0$ Yes

Deriv: $k_1 + 2k_2 x - \frac{k_3}{x^2} = 0 \longrightarrow \boxed{k_1 = 0}$

2nd Deriv: $2k_2 + \frac{2k_3}{x^3} = 0 \longrightarrow \boxed{k_2 = 0}$

3rd Deriv: $-6k_3/x^4 = 0$ $x = 1 : \boxed{k_3 = 0}$