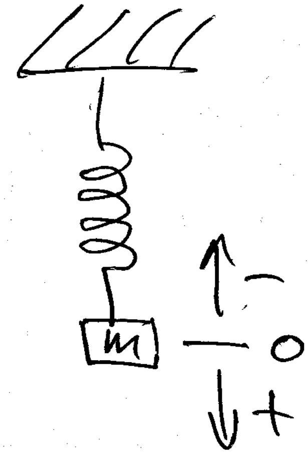
Laplace transforms

Forcing functions

- second order linear equations w/const coeffs model many kinds of oscillatory motion
- mass + spring systems.
- electrical circuits

Mass + Spring systems

$$\underset{\text{mass}}{m}u'' + \underset{\text{damping const}}{\gamma}u' + \underset{\text{spring const.}}{k}u = \underset{\text{forcing function}}{F}$$



$u(t)$ = displacement at time t.

A. Unforced systems.

1. undamped $\gamma = 0$.

   oscillatory motion $u(t) = R\cos(\omega_0 t - \delta)$

   ω_0 - natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$

   R, δ depend on initial conditions.

2. Damping $\gamma > 0$.

   a. small damping $0 < \gamma < 2\sqrt{mk}$

   damped oscillations

   b. critical or overdamped $\gamma \geq 2\sqrt{mk}$

   decaying solutions - no oscillation

B. Forced systems $F \neq 0$.

$\boxed{\text{Periodic forcing } F = F_0 \cos \omega t}$

1. Undamped $\gamma = 0$

a. $\omega = \omega_0 \rightarrow$ resonance

and $u(t) \rightarrow \infty$ as $t \rightarrow \infty$.

b. $\omega \neq \omega_0 \longrightarrow$ beat

2. Damped $\gamma > 0$

$$mu'' + \gamma u' + ku = F_0 \cos \omega t \qquad \omega \neq \omega_0 = \sqrt{\frac{k}{m}}$$

$$u(t) = R_1 e^{-\gamma t/2m} \cos(\mu t - \delta_1)$$

R_1, δ_1 depend on initial cond.

$\uparrow$ quasi-frequency.

$$+ \underbrace{(A \cos \omega t + B \sin \omega t)}_{R_2 \cos(\omega t - \delta_2)}$$

dies out as $t \to \infty$ (transient solution)

$R_2 \cos(\omega t - \delta_2) \leftarrow$ Persists (steady-state solution)

How does R_2 depend on ω and F_0?

After some work:

$$R_2 = \left(\frac{F_0}{k}\right)\left[\frac{1}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(\frac{\gamma^2}{mk}\right)\left(\frac{\omega}{\omega_0}\right)^2}\right]^{1/2}$$

If $\frac{\omega}{\omega_0}$ is very small then $[-] \approx 1$ and $R_2 \approx \frac{F_0}{k}$. $\frac{\omega}{\omega_0}$ small means forcing has low frequency (slow oscillations)

If $\frac{\omega}{\omega_0} \approx 1$ then $R_2 \approx \frac{F_0}{k} \cdot \frac{\sqrt{mk}}{\gamma}$ so oscillations are large if γ is small i.e. small damping

if $\frac{\omega}{\omega_0}$ very large then forcing has very fast oscillations and $[-]$ is very small so R_2 is very small.

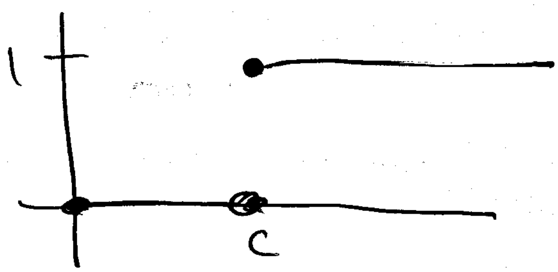
6.4 Discontinuous Forcing Functions

We have looked at periodic forcing.
What would discontinuous forcing look like?

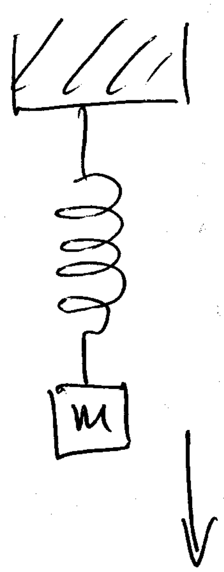
$\gamma = 0.$

$mu'' + \gamma u' + ku = g(t)$

$u(0) = 1, \quad u'(0) = 0$



$u_c(t) = g(t)$



discont forcing is like suddenly increasing gravity (or changing the mass, or turning on floor magnet, etc.)

(1) oscillatory motion for $0 \le t < c$

solving $mu'' + ku = 0, \quad u(0) = 1, \quad u'(0) = 0$

$$y_A(t) = R\cos(\omega_0 t - \delta) = \cos(\omega_0 t)$$

$R = 1$, $\delta = 0$

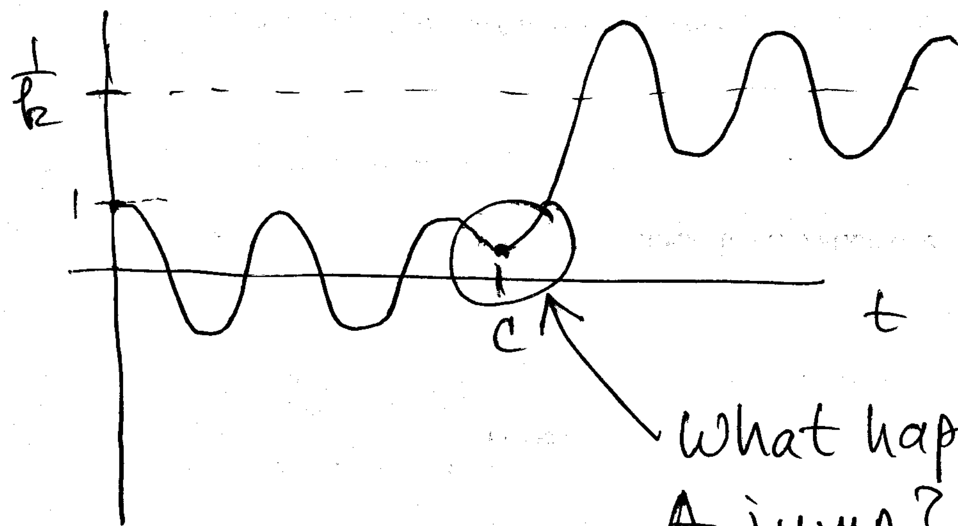
(2) at $t = c$ we are solving

$mu'' + ku = 1$ with new initial conditions

$u(c) = y_A(c) \quad u'(c) = y_A'(c)$

What will solution look like?

$$y_B(t) = \underbrace{R_1 \cos(\omega_0 t - \delta_1)}_{\text{homogeneous solution}} + \underbrace{\frac{1}{k}}_{\text{particular solution}}$$



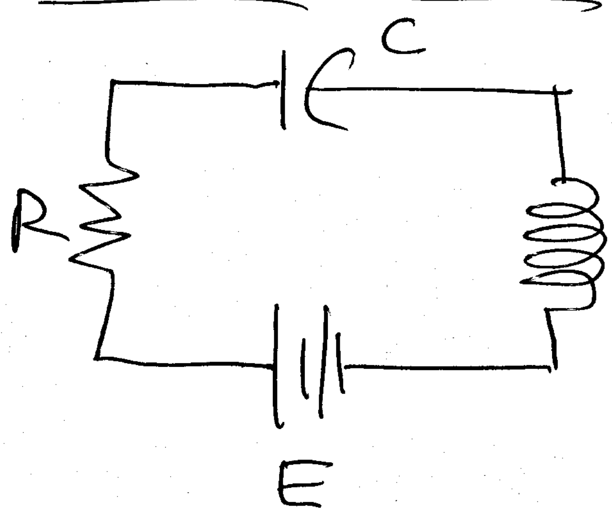
What happens at $t=c$?

A jump? NO

A corner? NO

A discontinuous second derivative? YES.

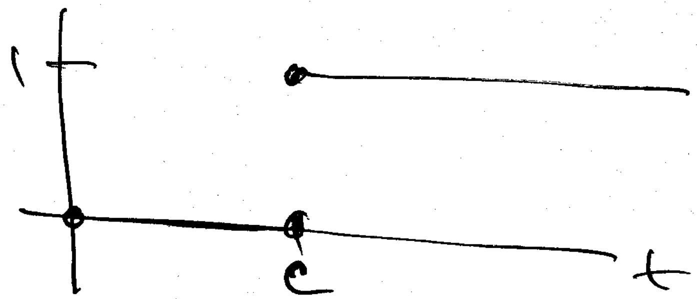
Electrical circuit



$$LQ'' + RQ' + \frac{1}{C}Q = E$$

Q – charge on capacitor

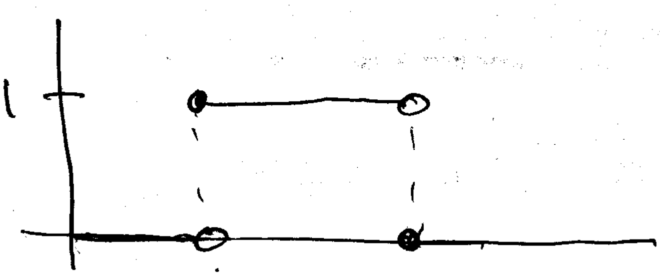
$E(t)$:



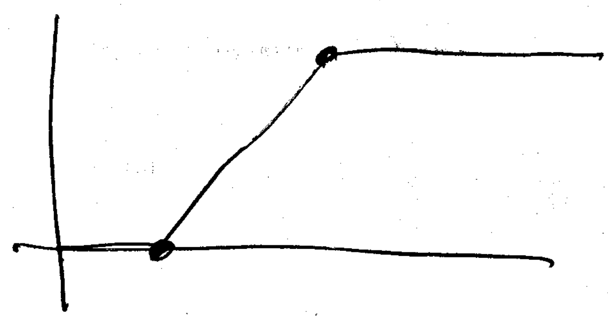
$= u_c(t)$

this is like flipping on a switch.

other examples:

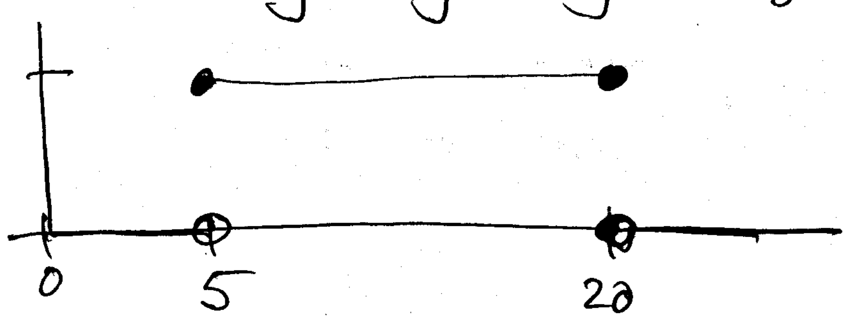


impulsive force



ramp loading

e.g. 1 $\quad 2y'' + y' + 2y = u_5(t) - u_{20}(t) = g(t)$



: $g(t)$

$y(0) = 0, \; y'(0) = 0$

$$2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{u_5\} - \mathcal{L}\{u_{20}\}$$

$$2\left(s^2\mathcal{L}\{y\} - s\cancel{y'(0)}^{\,0} - \cancel{y(0)}^{\,0}\right) + s\mathcal{L}\{y\} - \cancel{y(0)}^{\,0} + 2\mathcal{L}\{y\}$$

$$= \mathcal{L}\{u_5\} - \mathcal{L}\{u_{20}\}$$

$$(2s^2+s+2)\mathcal{L}\{y\} = \frac{1}{s}\left(e^{-5s} - e^{-20s}\right)$$

$$\mathcal{L}\{y\} = \frac{e^{-5s} - e^{-20s}}{s(2s^2+s+2)}$$

$$\frac{1}{s(2s^2+s+2)} = \frac{A}{s} + \frac{Bs+C}{2s^2+s+2}$$

$$= \frac{A(2s^2+s+2) + s(Bs+C)}{s(2s^2+s+2)}$$

$$A(2s^2+s+2) + s(Bs+C) = 1 \qquad s=0$$

$$2A = 1 \longrightarrow \underline{\underline{A = \tfrac{1}{2}}}$$

$$\frac{d}{ds}: A(4s+1) + 2Bs + C = 0 \qquad s=0$$

$$A + C = 0 \longrightarrow \underline{\underline{C = -\tfrac{1}{2}}}$$

$$\frac{d^2}{ds^2}: 4A + 2B = 0 \longrightarrow \underline{\underline{B = -1}}$$

$$\mathcal{L}\{y\} = \left(e^{-5s} - e^{-20s}\right)\left(\frac{1}{2}\cdot\frac{1}{s} - \frac{s+\frac{1}{2}}{2s^2+s+2}\right)$$

$$= \left(e^{-5s} - e^{-20s}\right)\left(\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\,\frac{s+\frac{1}{2}}{s^2+\frac{1}{2}s+1}\right)$$

$$= \frac{1}{2}\left(e^{-5s} - e^{-20s}\right)\left(\frac{1}{s} - \frac{\left(s+\frac{1}{4}\right)+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right)$$

$$= \frac{1}{2}\left(e^{-5s} - e^{-20s}\right)\left(\frac{1}{s} - \frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} - \frac{1}{4}\,\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right)$$

$$\mathcal{L}^{-1}\left(\text{———}\right) = 1 - e^{-t/4}\cos\left(\frac{\sqrt{15}}{4}t\right) - \frac{1}{\cancel{4}}\cdot\frac{\cancel{4}}{\sqrt{15}}\,e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

$$= h(t)$$

$$\mathcal{L}\{y\} = \frac{1}{2}\left(e^{-5s} - e^{-20s}\right)\mathcal{L}\{h(t)\}.$$