2.3 Modeling with 1st order equations

1. Mixture models.

e.g. 1. t = time in minutes

Q = amount of salt in tank in lbs

r = rate at which water is entering and leaving tank in $\frac{gal}{min}$

$$\frac{dQ}{dt} = \text{rate of salt in} - \text{rate of salt out}$$

$$= \frac{1}{4}\frac{lb}{gal}\cdot r\,\frac{gal}{min} - \frac{Q}{100}\frac{lb}{gal}\cdot r\,\frac{gal}{min}$$

$$= \frac{r}{4} - \frac{rQ}{100} = \frac{r(25-Q)}{100}$$

$$\frac{dQ}{25-Q} = \frac{r}{100}\,dt$$

$$-\ln|25-Q| = \frac{r}{100}t + c$$

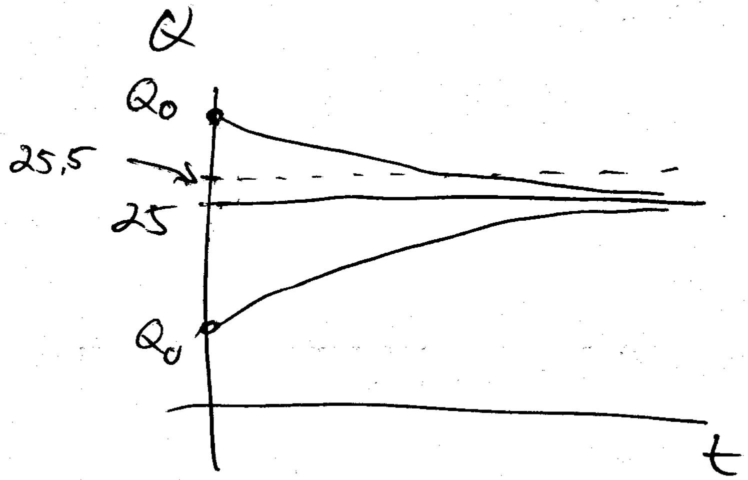
$$25-Q = c e^{-rt/100}$$

$$Q = 25 - c e^{-rt/100} \qquad \underline{Q(0) = Q_0}$$

$$Q_0 = 25 - c \rightarrow c = 25 - Q_0$$

$$Q = 25 - (25-Q_0)e^{-rt/100}.$$

(a) $Q_L = \lim_{t\to\infty} Q(t) = 25 \text{ lb}$ concentration $= \frac{25}{100} = \frac{1}{4} \frac{\text{lb}}{\text{gal}}$



(b) $Q_0 = 2Q_L = 50 \text{ lb}$ $r = 3 \text{ gal/min}$

Want T so that $Q(T) = (1.02)(25) = 25.5$

$$25.5 = 25 + 25 e^{-3T/100}$$

$$\frac{1}{50} = \frac{.5}{25} = e^{-3T/100} \rightarrow T = \frac{100}{3} \ln(50) \approx 130 \text{ min.}$$

(c) Solve: $25.5 = 25 + 25 e^{-r(45/100)}$

$$e^{-r(45/100)} = \frac{1}{50} \rightarrow r = \frac{100}{45} \ln(50)$$

$$\approx 8.7 \frac{\text{gal}}{\text{min}}.$$

e.g. 3

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

rate in: $\gamma(t)$ = concentration (g/gal) of chemical going in

rate of flow in = 5×10^6 gal/yr.

rate in = $(5\times10^6)\gamma(t)$ g/yr.

$= (5\times10^6)(2+\sin 2t)$

rate out:

concentration of chemical in lake $= \frac{Q}{10\times10^6}$ g/gal

rate of flow out = 5×10^6 gal/yr.

rate out $= (5\times10^6)\cdot\frac{Q}{10\times10^6} = \frac{Q}{2}$ g/yr.

$$\frac{dQ}{dt} = (5\times10^6)(2+\sin 2t) - \frac{Q}{2}.$$

$$\frac{dQ}{dt} + \frac{1}{2}Q = (5\times10^6)(2+\sin 2t)$$

Linear but not seperable, so use integrating factor. $\mu(t) = e^{1/2 t}$

$$\frac{d}{dt}(e^{1/2t}Q) = (5\times10^6)(2+\sin 2t)e^{1/2t}$$

$$e^{1/2t}Q = (5\times10^6)\left(\int 2e^{1/2t}\,dt + \int e^{1/2t}\sin 2t\,dt\right)$$

$$e^{1/2t}Q = (5\times10^6)(\underline{4e^{1/2t}} + \quad)$$

$$\int(\sin 2t)e^{1/2t}\,dt = \frac{e^{1/2t}}{\frac{1}{4}+4}\left(\tfrac{1}{2}\sin 2t - 2\cos 2t\right) + c$$

$a=\frac{1}{2}$ $b=2$

$$= e^{1/2t}\left(\tfrac{2}{17}\sin 2t - \tfrac{8}{17}\cos 2t\right) + c.$$

$$Q = (5\times10^6)\Big(\underset{4+}{}\tfrac{2}{17}\sin 2t - \tfrac{8}{17}\cos 2t + \underline{\underline{c}}\,e^{-1/2t}\Big)$$

$Q(0) = 0$

$$0 = (\cancel{5\times10^6})\,\cancel{(0-4\tfrac{8}{17}+c)}\,(4+0-\tfrac{8}{17}+c)$$

$$c = -\frac{60}{17}$$

$$Q = (5\times10^6)\left(4 + \tfrac{2}{17}\sin 2t - \tfrac{8}{17}\cos 2t - \tfrac{60}{17}e^{-1/2t}\right)$$

2. Compound Interest.

t = time in years
S = value of investment in \$
r = interest rate in %.

Note: rS = rate at which value is increasing in \$/year.

Think: Say $r = 5\%$, $S_0 = \$100$.
rS is an <u>approximation</u> because money will accumulate in jumps depending on frequency of compounding.

$$\frac{dS}{dt} = rS \qquad S(0) = S_0$$

$$S(t) = S_0 e^{rt}$$

Now assume deposits or withdrawals at a constant rate in \$/yr.

$$\frac{dS}{dt} = rS + k \qquad \begin{array}{l} k > 0 \text{ deposits} \\ k < 0 \text{ withdrawals.} \end{array}$$

$$S(t) = S_0 e^{rt} + \frac{k}{r}\left(e^{rt} - 1\right).$$

eg. $r = .08$

$k = 2000$ \$/yr.

$S_0 = 0$

$$S(t) = \frac{2000}{.08}(e^{.08t} - 1) = 25000(e^{.08t} - 1)$$

$$S(40) \cong \$588{,}000$$

Suppose. $S_0 = \$600{,}000$ (switch units to \$1000's.)

$r = .08$ $k = -30{,}000$ \$/yr.

$$S(t) = 600e^{.08t} + \frac{-30}{.08}(e^{.08t} - 1)$$

$$= \cancel{600e^{.08t} - 37.5(e^{.08t} - 1)}$$

$$= 600e^{.08t} - 375(e^{.08t} - 1)$$

$$= 225e^{.08t} + 375 \quad //$$

Find k so that balance does not grow.

$$S(t) = 600e^{.08t} - \frac{k}{.08}(e^{.08t} - 1)$$

$$= \left(600 - \frac{k}{.08}\right)e^{.08t} + \frac{k}{.08}.$$

$600 - \frac{k}{.08} = 0 \to k = 48$ can withdraw up to \$48000/year.