Final Exam – Tues 12/13 1^{30}–4^{15}
Review Session (Sam) M 12^{30}–2^{30}

$\vec{x}' = A\vec{x}$ A – $n\times n$ matrix
Find eigenvalues + eigenvectors of A
eigenvalues – $r_1, r_2, r_3, \ldots, r_n$ (assume distinct)
eigenvectors – $\vec{\xi}^{(1)}, \vec{\xi}^{(2)}, \ldots, \vec{\xi}^{(n)}$ (linearly indep.)
$\vec{x}(t) = c_1\vec{\xi}^{(1)}e^{r_1 t} + \cdots + c_n\vec{\xi}^{(n)}e^{r_n t}$ (general solution)

7.6 Complex Eigenvalues
(That is, complex roots of $\det(A-rI)=0$)
Since A will always be real, any complex root will come in complex conjugate pairs, ie. $\lambda+i\mu, \lambda-i\mu$.

e.g. $\vec{x}' = A\vec{x}$ $A = \begin{pmatrix} -\frac{1}{2} & 2 \\ -2 & -\frac{1}{2} \end{pmatrix}$

Eigenvalues

$$\begin{vmatrix} -\frac{1}{2}-r & 2 \\ -2 & -\frac{1}{2}-r \end{vmatrix} = \left(-\tfrac{1}{2}-r\right)^2+4 = \left(r+\tfrac{1}{2}\right)^2+4 = 0$$

$\left(r+\frac{1}{2}\right)^2 = -4$. $r+\frac{1}{2} = \pm 2i$ $r = -\frac{1}{2}+2i$
$r = -\frac{1}{2}-2i$

Eigenvectors.

$r = -\frac{1}{2} + 2i$: $\left(\begin{array}{ccc} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right) \to \left(\begin{array}{ccc} -2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$-2i \cdot x = 2$

$-2i x_1 + 2x_2 = 0 \to -2i x_1 = -2 \to x_1 = \frac{-2}{-2i} = -i$

$x_2 = 1$

$\vec{\xi}^{(1)} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

What about $\vec{\xi}^{(2)}$? $\quad -\frac{1}{2} - 2i = \overline{-\frac{1}{2} + 2i}$

$r = -\frac{1}{2} + 2i \qquad r_1 = \bar{r} \qquad \overline{\lambda + i\mu} = \lambda - i\mu$

Know: $(A - rI)\vec{\xi}^{(1)} = \vec{0}$ complex conjugate.

so $\to (\bar{A} - \bar{r}I)\overline{\vec{\xi}^{(1)}} = \vec{0}$

since A real $\to (A - \bar{r}I)\overline{\vec{\xi}^{(1)}} = \vec{0}$

$-\frac{1}{2} - 2i$

Conclusion: Eigenvector for $-\frac{1}{2} - 2i$ is $\overline{\vec{\xi}^{(1)}} = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$r = -\frac{1}{2} - 2i$

$\left(\begin{array}{ccc} 2i & 2 & 0 \\ -2 & 2i & 0 \end{array}\right) \to \left(\begin{array}{ccc} 2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$2i x_1 + 2x_2 = 0 \to 2i x_1 + 2 = 0 \to 2i x_1 = -2 \to x_1 = \frac{-2}{2i} = i$

$x_2 = 1$

So $\vec{\xi}^{(2)} = \begin{pmatrix} i \\ 1 \end{pmatrix}$ as expected.

<u>Fact</u>: Eigenvectors corresponding to eigenvalues $r = \lambda + i\mu$ and $r = \lambda - i\mu$ are conjugates of each other.

Solutions:

$$\vec{x}^{(1)}(t) = \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{(-\frac{1}{2}+2i)t}$$

$$= \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{-1/2 t} e^{2it} = e^{-t/2} \begin{pmatrix} -i \\ 1 \end{pmatrix} (\cos 2t + i \sin 2t)$$

$$= e^{-t/2} \begin{pmatrix} -i(\cos 2t + i \sin 2t) \\ \cos 2t + i \sin 2t \end{pmatrix} = e^{-t/2} \begin{pmatrix} -i \cos 2t + \sin 2t \\ i \sin 2t + \cos 2t \end{pmatrix}$$

$$= e^{-t/2} \left[\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} + i \begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix} \right]$$

What about $\vec{x}^{(2)}(t)$? What we <u>hope</u> to get is

$$\vec{x}^{(2)}(t) = e^{-t/2} \left[\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} - i \begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix} \right]$$

<u>And we do!</u>

$$\vec{x}^{(2)}(t) = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(-\frac{1}{2}-2i)t}$$

$$= e^{-t/2}\begin{pmatrix} i \\ 1 \end{pmatrix}(\cos 2t - i\sin 2t)$$

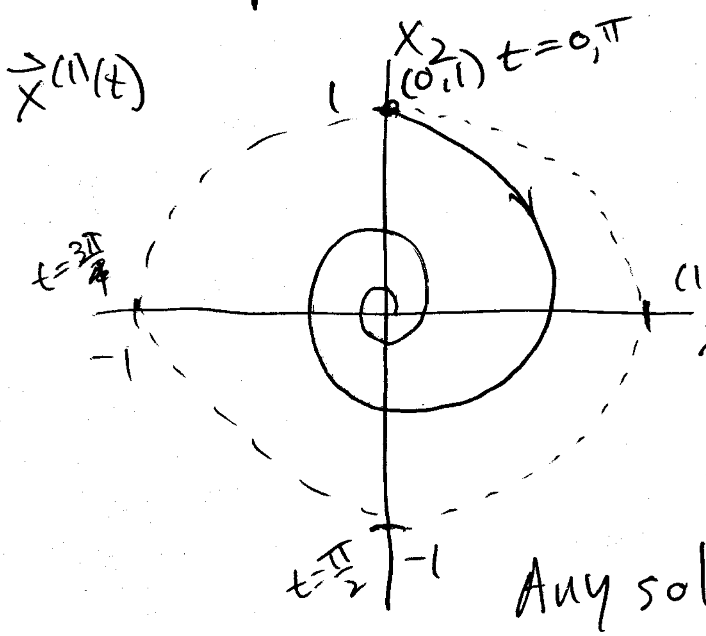
$$= e^{-t/2}\begin{pmatrix} i\cos 2t + \sin 2t \\ -i\sin 2t + \cos 2t \end{pmatrix}$$

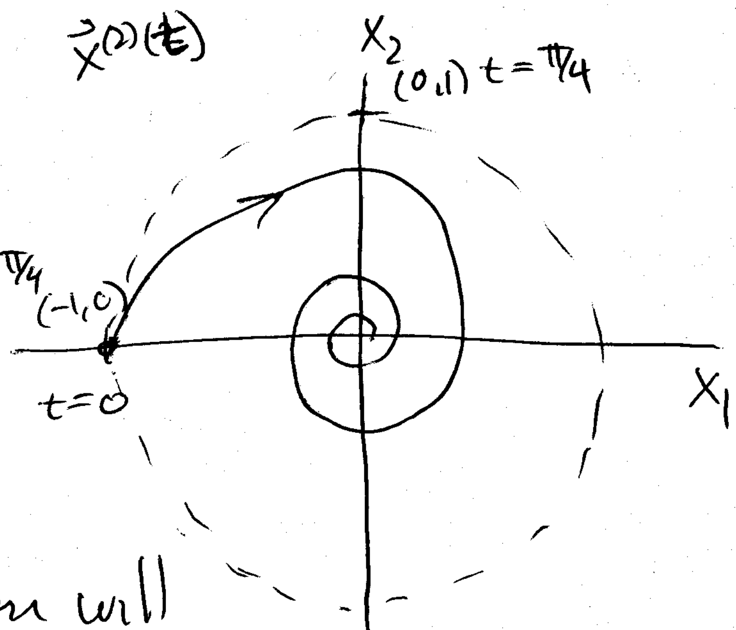
$$= e^{-t/2}\left[\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} - i\begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix}\right] \text{ as expected.}$$

We get two real solutions, linearly indep.

$$\vec{x}^{(1)}(t) = e^{-t/2}\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} \qquad \vec{x}^{(2)}(t) = e^{-t/2}\begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix}$$

Phase portrait:





Any solution will have same behavior

#6) $\vec{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \vec{x}$

Eigenvalues:

$$\cancel{\begin{vmatrix} 1-r & 2 \\ -5 & 1-r \end{vmatrix} = (1-r)^2 + 10 = (r-1)^2 + 10 = 0}$$

$$\begin{vmatrix} 1-r & 2 \\ -5 & -1-r \end{vmatrix} = (1-r)(-1-r) + 10 = (r-1)(r+1) + 10$$

$$= r^2 - 1 + 10 = r^2 + 9 = 0 \qquad r = 3i, \quad r = -3i$$

Eigenvectors:

$\underline{r = 3i}$

$$\left(\begin{array}{ccc} 1-3i & 2 & 0 \\ -5 & -1-3i & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc} 1-3i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$(1-3i)x_1 + 2x_2 = 0 \rightarrow (1-3i)x_1 = -2 \rightarrow x_1 = \frac{-2}{1-3i} \cdot \frac{1+3i}{1+3i}$$

$x_2 = 1$

$$= \frac{-2-6i}{10}$$

$$= \frac{-1-3i}{\cancel{10}\,5}$$

$$\vec{\xi}^{(1)} = \begin{pmatrix} -1-3i \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} + i \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

$$\boxed{\vec{\xi}^{(2)} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} - i\begin{pmatrix} -3 \\ 0 \end{pmatrix}} \leftarrow \text{not needed.}$$

$$\vec{\xi}^{(1)} e^{rt} = \left[\begin{pmatrix} -1 \\ 5 \end{pmatrix} + i\begin{pmatrix} -3 \\ 0 \end{pmatrix}\right] e^{3it}$$

$$= \left[\begin{pmatrix} -1 \\ 5 \end{pmatrix} + i\begin{pmatrix} -3 \\ 0 \end{pmatrix}\right] (\cos 3t + i \sin 3t)$$

$$= \left[\begin{pmatrix} -1 \\ 5 \end{pmatrix} \cos 3t - \begin{pmatrix} -3 \\ 0 \end{pmatrix} \sin 3t\right] + i\left[\begin{pmatrix} -3 \\ 0 \end{pmatrix} \cos 3t + \begin{pmatrix} -1 \\ 5 \end{pmatrix} \sin 3t\right]$$

$$= \begin{pmatrix} -\cos 3t + 3\sin 3t \\ 5\cos 3t \end{pmatrix} + i\begin{pmatrix} -3\cos 3t - \sin 3t \\ 5\sin 3t \end{pmatrix}$$

$$\underline{\underline{\vec{x}^{(1)}(t) = \begin{pmatrix} -\cos 3t + 3\sin 3t \\ 5\cos 3t \end{pmatrix}}} \qquad \underline{\underline{\vec{x}^{(2)}(t) = \begin{pmatrix} -3\cos 3t - \sin 3t \\ 5\sin 3t \end{pmatrix}}}$$

What about initial conditions?

<u>e.g.</u> $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with previous equation.

$$\vec{x}(t) = c_1\begin{pmatrix} -\cos 3t + 3\sin 3t \\ 5\cos 3t \end{pmatrix} + c_2\begin{pmatrix} -3\cos 3t - \sin 3t \\ 5\sin 3t \end{pmatrix}$$

$$\vec{x}(0) = c_1\begin{pmatrix} -1 \\ 5 \end{pmatrix} + c_2\begin{pmatrix} -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\therefore$ Solve $\begin{pmatrix} -1 & -3 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{array}{l} -c_1 - 3c_2 = 1 \\ 5c_1 = 1 \end{array}$

$\cancel{c_2 = \frac{1}{5} \to -c_1 - \frac{3}{5} = 1 \to c_1 = -\frac{8}{5}}$

$c_1 = \frac{1}{5} \longrightarrow -\frac{1}{5} - 3c_2 = 1$

$c_2 = -\frac{2}{5}$

$$\vec{x}(t) = \frac{1}{5}\begin{pmatrix} -\cos 3t + 3\sin 3t \\ 5\cos 3t \end{pmatrix} - \frac{2}{5}\begin{pmatrix} -3\cos 3t - \sin 3t \\ 5\sin 3t \end{pmatrix}$$

<u>e.g.</u> #(0) $\vec{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}\vec{x} \quad \vec{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

<u>Eigenvalues</u>

$$\begin{vmatrix} -3-r & 2 \\ -1 & -1-r \end{vmatrix} = (-3-r)(-1-r) + 2 = (r+3)(r+1) + 2$$

$$= r^2 + 4r + 5 \qquad r = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$$

Eigenvector

$r=-2+i$ $\begin{pmatrix} -1-i & 2 & 0 \\ -1 & 1-i & 0 \end{pmatrix} \to \begin{pmatrix} -1-i & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$(-1-i)x_1+2x_2=0 \to (-1-i)x_1+2(-1-i)=0$

$x_2=-1-i$ $x_1=-2$

$$\vec{\xi}^{(1)}=\begin{pmatrix}-2\\-1-i\end{pmatrix}=\begin{pmatrix}-2\\-1\end{pmatrix}+i\begin{pmatrix}0\\-1\end{pmatrix}$$

$$\vec{\xi}^{(1)}e^{rt}=\left[\begin{pmatrix}-2\\-1\end{pmatrix}+i\begin{pmatrix}0\\-1\end{pmatrix}\right]e^{-2t}(\cos t+i\sin t)$$

$$=e^{-2t}\left[\left(\begin{pmatrix}-2\\-1\end{pmatrix}\cos t-\begin{pmatrix}0\\-1\end{pmatrix}\sin t\right)+i\left(\begin{pmatrix}0\\-1\end{pmatrix}\cos t+\begin{pmatrix}-2\\-1\end{pmatrix}\sin t\right)\right]$$

$$\vec{x}^{(1)}(t)=e^{-2t}\begin{pmatrix}-2\cos t\\-\cos t+\sin t\end{pmatrix} \qquad \vec{x}^{(2)}(t)=e^{-2t}\begin{pmatrix}-2\sin t\\-\cos t-\sin t\end{pmatrix}$$

$$\vec{x}(0)=\begin{pmatrix}1\\-2\end{pmatrix}$$

$$\vec{x}(t)=c_1\begin{pmatrix}-2\\-1\end{pmatrix}+c_2\begin{pmatrix}0\\-1\end{pmatrix}=\begin{pmatrix}1\\-2\end{pmatrix}$$

Solve $-2c_1=1 \longrightarrow c_1=-\frac{1}{2}$

$-c_1-c_2=-2 \to \frac{1}{2}-c_2=-2 \to c_2=5/2$

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$$\vec{x}(t) = -\frac{1}{2} e^{-2t} \begin{pmatrix} -2\cos t \\ -\cos t + \sin t \end{pmatrix} + \frac{5}{2} e^{-2t} \begin{pmatrix} -2\sin t \\ -\cos t - \sin t \end{pmatrix}$$