Quiz 1 will be returned in RCT
Quizzes are out of 5 points.
Solutions to quizzes will be posted.
Quiz 2: sections 1.2, 1.3.

Looking at $\frac{dy}{dt} + p(t)y = g(t)$ 1st order linear equation

Idea: Find $\mu(t)$ so that our equation is equivalent to: $\underline{\frac{d}{dt}(\mu y) = \mu g}$.

$\mu \frac{dy}{dt} + \underline{\mu'} y = \mu g \qquad \mu \frac{dy}{dt} + \underline{\mu p} y = \mu g.$

Find μ satisfying $\frac{d\mu}{dt} = \mu(t) p(t)$

$$\int \frac{d\mu}{\mu(t)} = \int p(t)\,dt$$

$$\ln|\mu(t)| = \int p(t)\,dt$$

$$\underline{\mu(t) = e^{\int p(t)\,dt}}$$

eg 1 $$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}$$

$p(t) = \frac{1}{2}$ $\quad g(t) = \frac{1}{2}e^{t/3}$ $\qquad \int p(t)\,dt = \int \frac{1}{2}\,dt = \frac{1}{2}t$

(any antiderivative of p will do.)

$$\mu(t) = e^{\int p(t)\,dt} = e^{\frac{1}{2}t}$$

integrating factor.

$$\underbrace{e^{\frac{1}{2}t}\frac{dy}{dt} + \frac{1}{2}e^{\frac{1}{2}t}y}_{} = \frac{1}{2}e^{t/3}e^{t/2}$$

$$\frac{d}{dt}\left(e^{1/2 t}y\right) = \frac{1}{2}e^{5/6 t}$$

$$e^{\frac{1}{2}t}y = \frac{\cancel{6}^{\,3}}{5}\cdot\frac{1}{\cancel{2}}e^{5/6 t} + c$$

$$\boxed{y = \frac{3}{5}e^{1/3 t} + ce^{-1/2 t} \quad \text{general solution}}$$

Add an initial condition e.g. $y(0) = 1$.

$$y(0) = \frac{3}{5} + c = 1 \longrightarrow c = \frac{2}{5}$$

IVP

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3} \quad y(0) = 1$$

$$y = \frac{3}{5}e^{t/3} + \frac{2}{5}e^{-t/2}$$

eg 3 $\quad t\dfrac{dy}{dt} + 2y = 4t^2 \quad y(1) = 2 \quad$ (IVP)

(a) put equation in standard form.

$$\frac{dy}{dt} + \underbrace{\frac{2}{t}}_{p(t)}\, y = \underbrace{4t}_{g(t)}$$

(b) Find $\mu(t)$. $\quad p(t) = \frac{2}{t} \quad \int p(t)\,dt = \int \frac{2}{t}\,dt$

$$= 2\ln|t|$$

$$= \ln|t|^2 = \ln(t^2)$$

$$\mu(t) = e^{\ln(t^2)} = t^2$$

(c) Solve. $\quad t^2 \dfrac{dy}{dt} + t^2 \cdot \dfrac{2}{t} y = t^2 \cdot 4t$

$$t^2 \frac{dy}{dt} + 2ty = 4t^3$$

$$\frac{d}{dt}(t^2 y) = 4t^3$$

$$t^2 y = t^4 + c$$

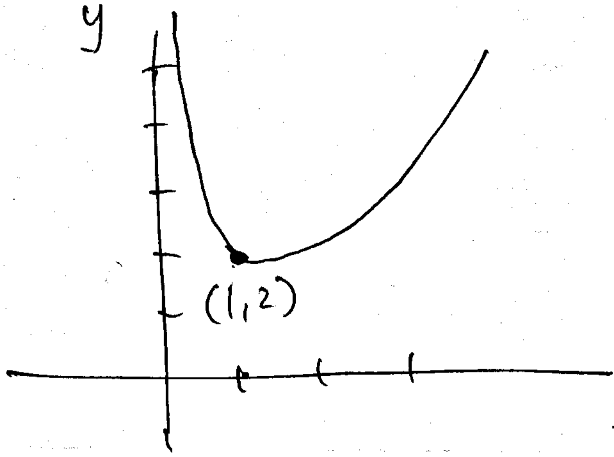
$$y = t^2 + \frac{c}{t^2} \qquad \text{general solution}$$

(d) Find c. $\quad y(1) = 2$

$$y(1) = 1 + c = 2 \longrightarrow c = 1$$

$$y = t^2 + \frac{1}{t^2} \quad //$$

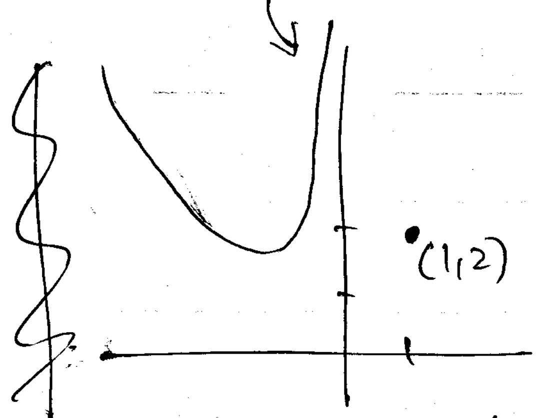
Q: For which t is the solution valid?



Solution is valid for $t>0$ only.

$y = t^2 + \frac{1}{t^2}, t<0.$

Why isn't this a solution?



Because it does not pass through $(1,2)$.

If the IVP were $t\frac{dy}{dt} + 2y = 4t^2$ $\underline{y(-1)=2}$

then this would be a solution.

2.2 Separable Equations.

Our equation: $\frac{dy}{dt} + ay = b$

Separate variables:

$$(ay - b) + \frac{dy}{dt} = 0$$

$$a\left(y - \frac{b}{a}\right) + \frac{dy}{dt} = 0$$

$$a + \frac{dy/dt}{y - b/a} = 0$$

$$\int a \, dt + \int \frac{dy}{y - b/a} = 0$$

$$at + \ln|y - b/a| = c$$

$$\ln|y - b/a| = -at + c$$

$$|y - b/a| = e^{c} e^{-at}$$

$$y - b/a = c e^{-at} \longrightarrow y = \frac{b}{a} + c e^{-at}$$

In general we can solve equations of the form

$$M(x) + N(y) \frac{dy}{dx} = 0$$

Note that we have switched from t to x. Get used to it.

eg $\dfrac{dy}{dx} = \dfrac{x^2}{1-y^2}$

$$(1-y^2)\frac{dy}{dx} = x^2$$

$$\underbrace{-x^2}_{M(x)} + \underbrace{(1-y^2)}_{N(y)}\frac{dy}{dx} = 0$$

$$-x^2\,dx + (1-y^2)\,dy = 0$$

integrate

$$-\frac{1}{3}x^3 + y - \frac{1}{3}y^3 = c$$ ← solution to the ODE as an implicitly defined curve.

Suppose I want the solution through $y(1)=1$. Try to solve for c. $x=1$, $y=1$

$$-\frac{1}{3} + 1 - \frac{1}{3} = c \longrightarrow c = \frac{1}{3}$$

$-\frac{1}{3}x^3 + y - \frac{1}{3}y^3 = \frac{1}{3}$. This solution is valid for all x.

(Look at graph on p. 43)

General solution:

$$M(x) + N(y)\frac{dy}{dx} = 0$$

$$M(x)\,dx + N(y)\,dy = 0$$

$$\underbrace{\int M(x)\,dx}_{\substack{\text{some antiderivative}\\ \text{of } M,\\ \text{call it } H_1(x)}} + \underbrace{\int N(y)\,dy}_{\substack{\text{some antiderivative}\\ \text{of } N, \text{ call it}\\ H_2(y)}} = c$$

Solution has form $\boxed{H_1(x) + H_2(y) = c.}$

eg. 2 $\quad \dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)} \qquad y(0) = -1.$

$$(3x^2+4x+2) - 2(y-1)\frac{dy}{dx} = 0$$

$$(3x^2+4x+2)\,dx = 2(y-1)\,dy$$

$$x^3+2x^2+2x+c = y^2-2y \qquad y(0) = -1$$

$$0 + c = 1+2 = 3 \qquad \underline{c=3}$$

$$y^2-2y = x^3+2x^2+2x+3 \leftarrow$$ In this case I can solve for y explicitly

$y^2-2y+1 = x^3+2x^2+2x+4 = (x^2+2)(x+2)$

$(y-1)^2 = (x^2+2)(x+2)$

$y-1 = \pm (x^2+2)^{1/2}(x+2)^{1/2}$

$y = 1 \pm (x^2+2)^{1/2}(x+2)^{1/2}$. Do I want + or − ?

Since $y(0)=-1$, $y = 1-(x^2+2)^{1/2}(x+2)^{1/2}$ //

For which x is this solution valid?

Since we must have $x+2 \geq 0$ this solution only holds for $x \geq -2$.

One more adjustment: Since $x=-2$ corresponds to $y=1$, the solution above is valid only for $x > -2$. because $y=1$ corresponds to vertical tangent.

#28) $y' = \dfrac{t\,y(4-y)}{1+t}$ $\quad y(0) = y_0 > 0.$

Equilibrium solutions: $y = 0 \quad y = 4$