No class Tuesday

3.6 Variation of Parameters.

Idea: $y'' + p(t)y' + q(t)y = g(t)$

1) Solve homogeneous equation
$$y'' + p(t)y' + q(t)y = 0$$
Fundamental set of solutions $y_1(t), y_2(t)$.

2) Find a particular solution of non-homogeneous equation, $Y(t)$.

3) General solution: $y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$

Undetermined coefficients: If $g(t)$ has a particular form and equation has const. coeffs then you can guess at $Y(t)$.

Variation of parameters: You can find $Y(t)$ for any $g(t)$ but you need to know $y_1(t), y_2(t)$.

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e.g. $y'' + 4y = 3\csc(t)$

1) Homogeneous equation
$$r^2 + 4 = 0 \longrightarrow r = \pm 2i$$
$$y_1(t) = \cos(2t), \quad y_2(t) = \sin(2t)$$

2) Non-homogeneous equation

$$Y(t) = u_1(t)\cos(2t) + u_2(t)\sin(2t)$$

$$Y'(t) = -2u_1(t)\sin(2t) + u_1'(t)\cos(2t) + 2u_2(t)\cos(2t)$$
$$+ u_2'(t)\sin(2t).$$

$Y''(t) =$ ~ 8 terms!

We will end up with one equation and 2 unknowns, u_1, u_2.

so we add an equation arbitrarily (not really)

Assume $\boxed{u_1'(t)\cos(2t) + u_2'(t)\sin(2t) = 0.}$

Now $Y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t)$

$$Y''(t) = -4u_1(t)\cos(2t) - 2u_1'(t)\sin(2t) - 4u_2(t)\sin(2t)$$
$$+ 2u_2'(t)\cos(2t).$$

Plug in to original.

$$-4u_1\cancel{\cos(2t)} - 2u_1'\sin(2t) - 4u_2\cancel{\sin(2t)} + 2u_2'\cos(2t)$$
$$+ \cancel{4u_1\cos(2t)} + \cancel{4u_2\sin(2t)} = 3\csc(t)$$

Leaves us with 2 equations in 2 unknowns u_1', u_2'.

$$(u_1' \cos(2t) + u_2' \sin(2t) = 0)\, 2\sin(2t)$$
$$(-2u_1' \sin(2t) + 2u_2' \cos(2t) = 3\csc(t))\, \cos(2t)$$

$$2u_2' \sin^2(2t) + 2u_2' \cos^2(2t) = 3\csc(t)\cos(2t)$$

$$\longrightarrow \underline{u_2' = \frac{3}{2}\csc(t)\cos(2t)}$$

$$u_1' = -u_2' \frac{\sin(2t)}{\cos(2t)} = -\frac{3}{2}\csc(t)\cancel{\cos(2t)} \cdot \frac{\sin(2t)}{\cancel{\cos(2t)}}$$
$$\underline{= -\frac{3}{2}\csc(t)\sin(2t).}$$

Now we need to integrate:

$\csc(t) = \frac{1}{\sin(t)}$ $\quad$ $\cos(2t) = \cos^2(t) - \sin^2(t) = 1 - 2\sin^2(t)$
$\quad$ $\sin(2t) = 2\sin(t)\cos(t)$

$$u_1' = -\frac{3}{\cancel{2}}\frac{1}{\cancel{\sin(t)}} \cdot \cancel{2}\cancel{\sin(t)}\cos(t) = -3\cos(t)$$
$$\boxed{u_1 = -3\sin(t)}$$
$$u_2' = \frac{3}{2}\frac{1}{\sin(t)}(1 - 2\sin^2(t)) = \frac{3}{2}\csc(t) - 3\sin(t)$$
$$\boxed{u_2 = -\frac{3}{2}\ln|\csc(t) + \cot(t)| + 3\cos(t)}$$

etc....

Do we always get the nice cancellation?

YES. General case $y''+p(t)y'+q(t)y=g(t)$

Assume we have y_1, y_2 homogeneous solutions

look for $Y = u_1 y_1 + u_2 y_2$.

$Y' = u_1 y_1' + \boxed{u_1' y_1 + u_2' y_2} + u_2 y_2'$

Assume $\boxed{u_1' y_1 + u_2' y_2 = 0}$

$Y' = u_1 y_1' + u_2 y_2'$

$Y'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'$

Plug in:

$u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2' + p u_1 y_1' + p u_2 y_2'$

$\quad + q u_1 y_1 + q u_2 y_2 = g.$

$u_1 [\underbrace{y_1'' + p y_1' + q y_1}_{0}] + u_2 [\underbrace{y_2'' + p y_2' + q y_2}_{0}]$

$+ u_1' y_1' + u_2' y_2' = g$

Recall: y_1, y_2 solved homogeneous equation

$\boxed{u_1' y_1' + u_2' y_2' = g}$

Solve the system: $\boxed{\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g \end{aligned}}$

Write as matrix equation.

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \frac{1}{ad-bc} \leftarrow$$

In our case! $y_1 y_2' - y_2 y_1' = W(y_1, y_2)$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W(y_1,y_2)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix} = \frac{1}{W(y_1,y_2)} \begin{bmatrix} -y_2 g \\ y_1 g \end{bmatrix}$$

$$\therefore \quad u_1' = \frac{-y_2 g}{W(y_1,y_2)} \qquad u_2' = \frac{y_1 g}{W(y_1,y_2)}$$

$$u_1 = -\int \frac{y_2 g}{W(y_1,y_2)} \qquad u_2 = \int \frac{y_1 g}{W(y_1,y_2)}$$

$$\boxed{Y(t) = -y_1 \int \frac{y_2 g}{W(y_1,y_2)} + y_2 \int \frac{y_1 g}{W(y_1,y_2)}}$$

e.g. #2) $y'' - y' - 2y = 2e^{-t}$

1) Homogeneous solutions

$r^2 - r - 2 = 0$
$(r+1)(r-2) = 0$
$r = -1,\ r = 2$ $\rightarrow$ $y_1 = e^{-t}$, $y_2 = e^{2t}$

2) Set $Y = u_1 e^{-t} + u_2 e^{2t}$

Solve the system: $u_1' e^{-t} + u_2' e^{2t} = 0$
$-u_1' e^{-t} + 2u_2' e^{2t} = 2e^{-t}$

$3u_2' e^{2t} = 2e^{-t}$
$u_2' = \frac{2}{3} e^{-3t}$

$u_1' = -u_2' e^{2t} \cdot e^{t}$
$= -u_2' e^{3t}$
$= -\frac{2}{3} e^{-3t} e^{3t} = -\frac{2}{3}$.

$u_1 = -\frac{2}{3} t \quad u_2 = -\frac{2}{9} e^{-3t}$

$Y(t) = -\frac{2}{3} t e^{-t} - \frac{2}{9} e^{-3t} e^{2t}$
$= -\frac{2}{3} t e^{-t} - \frac{2}{9} e^{-t}$ solution to homogeneous equ. so I dont need it.

Undetermined coefficients. $Y(t) = Ae^{-t}$ no good since e^{-t} is a homog. solution

So try $Y(t) = Ate^{-t}$
$Y'(t) = -Ate^{-t} + Ae^{-t}$
$Y''(t) = Ate^{-t} - Ae^{-t} - Ae^{-t} = Ate^{-t} - 2Ae^{-t}$

Plug in

$$\cancel{Ate^{-t}} - 2Ae^{-t} + \cancel{Ate^{-t}} - Ae^{-t} - \cancel{2Ate^{-t}} = 2e^{-t}$$

$$-3Ae^{-t} = 2e^{-t} \longrightarrow A = -\frac{2}{3}.$$

$\therefore Y(t) = -\frac{2}{3}te^{-t}$ why the difference?

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eg #10) $y'' - 2y' + y = \frac{e^t}{1+t^2}$

1) Homogeneous:

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$

$r = +1$

$y_1 = e^{+t}$

$y_2 = te^{+t}$

2) $Y = u_1 e^t + u_2 t e^t$

System: $u_1' e^t + u_2' t e^t = 0$

$$-\left(u_1' e^t + u_2'(te^t + e^t) = \frac{e^t}{1+t^2}\right)$$

$$u_2' \cancel{te^t} - u_2'(\cancel{te^t} + e^t) = \frac{-e^t}{1+t^2}$$

$$-u_2' e^t = \frac{-e^t}{1+t^2} \longrightarrow u_2' = \frac{1}{1+t^2} \longrightarrow \underline{u_2 = \tan^{-1}(t)}.$$

$$u_1' = -u_2' t e^t e^{-t} = -u_2' t$$

$$u_1' = \frac{1}{2}\left(\frac{-2t}{1+t^2}\right) \longrightarrow \underline{u_1 = -\frac{1}{2}\ln(1+t^2)}$$

$$Y(t) = -\frac{1}{2}\ln(1+t^2)e^t + te^t\tan^{-1}(t).$$ //

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eg #16) $(1-t)y'' + ty' - y = 2(t-1)^2 e^{-t} \quad 0<t<1$

Given $y_1 = e^t$, $y_2 = t$.

$$Y(t) = u_1 e^t + u_2 t$$

Standard form: $y'' + \frac{t}{1-t}y' - \frac{1}{1-t}y = \underbrace{-2(t-1)e^{-t}}_{g(t)}$

System: $u_1' e^t + u_2' t = 0$

$$-\left(u_1' e^t + u_2' = -2(t-1)e^{-t}\right)$$

$$u_2'(\cancel{t-1}) = +2(\cancel{t-1})e^{-t}$$

$$u_2' = 2e^{-t} \longrightarrow \underline{u_2 = -2e^{-t}}$$

$$u_1' = -u_2' t e^{-t} = -2e^{-t}\cdot te^{-t} = -2te^{-2t}$$

$$\longrightarrow u_1 = -2\left(-\frac{1}{2}te^{-2t} + \frac{1}{2}\int e^{-2t}dt\right)$$

$$= \underline{te^{-2t} + \frac{1}{2}e^{-2t}}$$

$$Y(t) = te^{-2t}e^t + \frac{1}{2}e^{-2t}e^t - 2te^{-t}$$

$$= te^{-t} + \frac{1}{2}e^{-t} - 2te^{-t} = \frac{1}{2}e^{-t} - te^{-t}$$ //