Final Exam 12/13 $1^{30} - 4^{15}$

Review Session Monday 12/12 $12^{30} - 2^{30}$

Room TBA (check website).

Quiz 14 7.1, 7.3.

7.5 Homogeneous Linear Systems with Constant Coefficients.

Solving

$$\vec{x}' = A\vec{x} \qquad A = n \times n \text{ constant matrix}$$

1. Look for solutions of the form

$$\vec{x} = \vec{\xi} e^{rt} \qquad \vec{\xi} = \text{some constant vector.}$$

$$\vec{x}' = \vec{\xi} r e^{rt} \longrightarrow \underbrace{\vec{\xi} r}\cancel{e^{rt}} = \underbrace{A\vec{\xi}}\cancel{e^{rt}}$$

$\vec{\xi}$ must satisfy $A\vec{\xi} = r\vec{\xi}$ ← r is an eigenvalue and $\vec{\xi}$ an eigenvector of A.

Procdure:

(a) Find eigenvalues of A and all corresponding eigenvectors. Two possibilities:

1) Find n distinct eigenvalues and hence always n distinct eigenvectors. These eigenvectors are always linearly independent. The eigenvalues may be complex.

2) Fewer than n distinct eigenvalues, i.e. repeated roots.

(We will look at case 1) only.)

(b) Eigenvalues are $r_1, r_2, \dots, r_n$ with corresponding eigenvectors $\vec{\xi}^{(1)}, \vec{\xi}^{(2)}, \dots, \vec{\xi}^{(n)}$

The general solution is:

$$\vec{x}(t) = c_1 \underline{\vec{\xi}^{(1)} e^{r_1 t}} + c_2 \underline{\vec{\xi}^{(2)} e^{r_2 t}} + \dots + c_n \underline{\vec{\xi}^{(n)} e^{r_n t}}$$

(c) What about Wronskian?

$$W\left[\vec{\xi}^{(1)}e^{r_1 t}, \ldots, \vec{\xi}^{(n)}e^{r_n t}\right]$$

$$= \begin{vmatrix} | & | & & | \\ \vec{\xi}^{(1)}e^{r_1 t} & \vec{\xi}^{(2)}e^{r_2 t} & \cdots & \vec{\xi}^{(n)}e^{r_n t} \\ | & | & & | \end{vmatrix}$$

$$= \underbrace{e^{r_1 t}e^{r_2 t}\cdots e^{r_n t}}_{\text{never zero for any } t.} \underbrace{\begin{vmatrix} | & | & & | \\ \vec{\xi}^{(1)} & \vec{\xi}^{(2)} & \cdots & \vec{\xi}^{(n)} \\ | & | & & | \end{vmatrix}}_{\neq 0 \text{ if and only if eigenvectors are linearly independent. Which they will always be in this case}}$$

Bottom line: Don't worry about Wronskian!

(d) Initial conditions

will have the form $\vec{x}(0) = \vec{x}_0$

we need:

$$\vec{x}_0 = c_1 \vec{\xi}^{(1)} + c_2 \vec{\xi}^{(2)} + \cdots + c_n \vec{\xi}^{(n)}$$

This is the system

$$\begin{pmatrix} | & | & & | \\ \vec{\xi}^{(1)} & \vec{\xi}^{(2)} & \cdots & \vec{\xi}^{(n)} \\ | & | & & | \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \vec{x}_0$$

which will always have <u>unique</u> solution.

<u>e.g. 1.</u> $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}$

$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ $\quad A - rI = \begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix}$

$$\det(A - rI) = \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)^2 - 4 = 0$$

$1 - r = \pm 2 \rightarrow \underline{r = -1, \ r = 3,}$ ← eigenvalues.

Eigenvectors:

$\underline{r=-1}$: Solve $(A+I)\vec{x}=\vec{0}$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2\begin{pmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$2x_1+x_2=0$ $\quad$ $x_2=t$ $\quad$ $\vec{\xi}^{(1)}=t\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

$x_1=-\frac{1}{2}t$ $\quad$ $t\in\mathbb{R}$

Take $\vec{\xi}^{(1)}=\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (i.e. $t=2$)

$\underline{r=3}$ $\begin{pmatrix} -2 & 1 & 0 \\ 4 & -2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

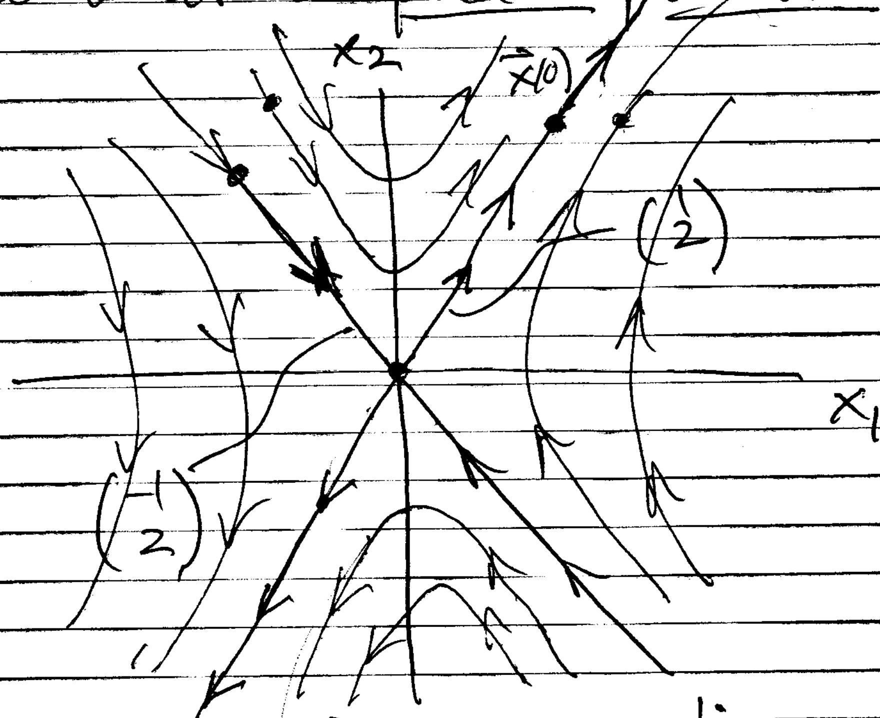
$-2x_1+x_2=0 \longrightarrow x_1=\frac{1}{2}x_2$ $\quad$ $\vec{\xi}^{(2)}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad$ $\underline{\underline{t=2}}$

$x_2=t$

General solution

$$\vec{x}(t) = c_1\begin{pmatrix}-1\\2\end{pmatrix}e^{-t} + c_2\begin{pmatrix}1\\2\end{pmatrix}e^{3t}$$

Look at a phase picture.



Suppose $\vec{x}(0)$ is on line parallel to $\begin{pmatrix}1\\2\end{pmatrix}$

$\vec{x}(0) = \alpha\begin{pmatrix}1\\2\end{pmatrix}$ some $\alpha \in \mathbb{R}$.

To find c_1, c_2 need to solve:

$$\begin{pmatrix}-1 & 1\\2 & 2\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}\alpha\\2\alpha\end{pmatrix} \longrightarrow \begin{matrix}c_1 = 0\\c_2 = \alpha\end{matrix}$$

So $\vec{x}(t) = \alpha\begin{pmatrix}1\\2\end{pmatrix}e^{3t}$

If $\vec{x}(0)$ is on line parallel to $\begin{pmatrix}-1\\2\end{pmatrix}$

$$\vec{x}(0) = \beta\begin{pmatrix}-1\\2\end{pmatrix} \qquad \begin{pmatrix}-1 & 1\\2 & 2\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}-\beta\\2\beta\end{pmatrix}$$

$$c_2 = 0 \qquad c_1 = \beta \qquad \vec{x}(t) = \beta\begin{pmatrix}-1\\2\end{pmatrix}e^{-t}$$

eg #16) $\vec{x}' = \begin{pmatrix}-2 & 1\\-5 & 4\end{pmatrix}\vec{x} \qquad \vec{x}(0) = \begin{pmatrix}1\\3\end{pmatrix}$

Eigenvalues

$$\begin{vmatrix}-2-r & 1\\-5 & 4-r\end{vmatrix} = (-2-r)(4-r)+5$$

$$= r^2-2r-3 = 0 \qquad (r+1)(r-3) = 0$$

$r=-1,\ r=3$

Eigenvectors:

$\underline{r=-1}$ $\left(\begin{array}{cc|c}-1 & 1 & 0\\-5 & 5 & 0\end{array}\right) \rightarrow \left(\begin{array}{cc|c}-1 & 1 & 0\\0 & 0 & 0\end{array}\right)$

$-x_1+x_2=0$

$x_1 = 1 \qquad \vec{\xi}^{(1)} = \begin{pmatrix}1\\1\end{pmatrix}$

$x_2 = 1 \leftarrow$ arbitrary.

$\underline{r=3}$ $\begin{pmatrix} -5 & 1 & 0 \\ -5 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -5 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$-5x_1 + x_2 = 0$ $x_1 = \frac{1}{5}$ $\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

$x_2 = 1$

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$$

Solve $\begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$\begin{aligned} c_1 + c_2 &= 1 \\ -(c_1 + 5c_2 &= 3) \\ \hline -4c_2 &= -2 \end{aligned}$$

$c_2 = \frac{1}{2}$ $c_1 = \frac{1}{2}$

$$\vec{x}(t) = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t} \quad //$$

e.g. #18)

$$\vec{x}' = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} -r & 0 & -1 \\ 2 & -r & 0 \\ -1 & 2 & 4-r \end{vmatrix} = -r\big(-r(4-r)\big) - (4-r) = 0$$

$(4-r)(r^2-1) = 0$ $\quad \boxed{r=4,\ r=1,\ r=-1}$

$r=4$ $\quad -\frac{1}{4}$

$$\begin{pmatrix} -4 & 0 & -1 & 0 \\ 2 & -4 & 0 & 0 \\ -1 & 2 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} -4 & 0 & -1 & 0 \\ 0 & -4 & -1/2 & 0 \\ 0 & 2 & 1/4 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} -4 & 0 & -1 & 0 \\ 0 & -4 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 = -1/4$
$x_2 = -\frac{1}{8}$
$x_3 = 1$

$-4x_2 - 1/2\, x_3 = 0$
$-4x_2 - 1/2 = 0$
$x_2 = -\frac{1}{8}$

$$\vec{\xi}^{(1)} = \begin{pmatrix} -2 \\ -1 \\ 8 \end{pmatrix}$$

etc....