Quiz 6 - Section 3.3

Finishing up 3.4 (Repeated Roots)

Reduction of Order:

1. $ay''+by'+cy=0$ $\rightarrow$ $y_1 = e^{r_1 t}$

$ar^2+br+c=0$ Seen: $y_2 = te^{r_1 t}$

$a(r-r_1)^2=0$

Why? Look for $y_2 = v(t)y_1(t)$
$= v(t)e^{r_1 t}$

substitute into $ay''+by'+cy=0$

Always get: $v''=0$

so $v(t)=t$ is a solution

2. $y''+p(t)y'+q(t)y=0$

If we know a solution y_1, we can find y_2 by:

(a) Assume $y_2(t)=v(t)y_1(t)$

(b) substitute into $y''+p(t)y'+q(t)y=0$

(c) Always get: $\boxed{y_1v''+(2y_1'+py_1)v'=0}$

This is a first order equation in $u=v'$.

e.g. $2t^2y''+3ty'-y=0,\ t>0$ $\quad y_1=t^{-1}$

Find y_2. Last time we did 2(a), 2(b) and got

$2tv''-v'=0$.

Could also use 2(c): $y_1=t^{-1}$, $p=\frac{3}{2t}$ so v satisfies

$y''+\frac{3t}{2t^2}y'-\frac{1}{2t^2}y=0$

$p(t)=\frac{3}{2t}$

$t^{-1}v''+(-2t^{-2}+\frac{3}{2t}t^{-1})v'=0$

$t^{-1}v''+(-2t^{-2}+\frac{3}{2}t^{-2})v'=0$

$(t^{-1}v''-\frac{1}{2}t^{-2}v'=0)\ 2t^2$

$2tv''-v'=0$.

3.5 Nonhomogeneous equations (Undetermined coefficients)

Now solving $y'' + p(t)y' + q(t)y = \underset{\neq 0}{g(t)}$.

or $L[y] = g(t)$.

<u>Idea</u>: 1. We know that <u>any</u> solution to $L[y]=0$ has the form $y = c_1 y_1 + c_2 y_2$ where y_1, y_2 is a fundamental set of solutions.

2. If we can find <u>any</u> solution to $L[y] = g$ (by any means whatsoever) call it $Y(t)$ then the general solution is $y = c_1 y_1 + c_2 y_2 + Y$. Y is called a <u>particular solution</u>.

3. Why? If Y_0 is some other solution to $L[y]=g$ then $Y_0 - Y$ satisfies:

$$L[Y_0 - Y] = L[Y_0] - L[Y] = g - g = 0$$

So $Y_0 - Y$ solves the homogeneous equation

Therefore $Y_0 - Y = c_1 y_1 + c_2 y_2$ so $\underline{Y_0 = c_1 y_1 + c_2 y_2 + Y}$

4. Finding a particular solution is the hard part!

e.g. $y'' - 3y' - 4y = 3e^{2t}$

(nonhomogeneous, linear, const. coeffs).
Want to find a particular solution $Y(t)$.
Guess: e^{2t} does not really change when differentiated so maybe $Y(t)$ has the form $Y(t) = Ae^{2t}$. Try it:

$Y'(t) = 2Ae^{2t}$, $Y''(t) = 4Ae^{2t}$ so

$$\cancel{4Ae^{2t}} - 6Ae^{2t} - \cancel{4Ae^{2t}} = 3e^{2t}$$

$-6A = 3 \Rightarrow A = -\frac{1}{2}$. So $\underline{Y(t) = -\frac{1}{2}e^{2t}}$

For general solution, solve $y'' - 3y' - 4y = 0$.

$r^2 - 3r - 4 = 0$ $\qquad y_1 = e^{-t}$

$(r+1)(r-4) = 0$ $\qquad y_2 = e^{4t}$

$r = -1, r = 4$

General solution: $y = c_1 e^{-t} + c_2 e^{4t} - \frac{1}{2}e^{2t}$

e.g. $y'' - 3y' - 4y = 2\sin t$

Try $Y(t) = A\sin t + B\cos t$

$$Y'(t) = A\cos t - B\sin t$$

$$Y''(t) = -A\sin t - B\cos t$$

Substitute:

$$-A\sin t - B\cos t - 3A\cos t + 3B\sin t - 4A\sin t - 4B\cos t = 2\sin t$$

$$(-A + 3B - 4A)\sin t + (-B - 3A - 4B)\cos t = 2\sin t$$

$$(-5A + 3B)\sin t + (-3A - 5B)\cos t = 2\sin t$$

$$-5A + 3B = 2 \longrightarrow \tfrac{25}{3}B + \tfrac{9}{3}B = 2$$

$$-3A + 5B = 0 \longrightarrow A = -\tfrac{5}{3}B$$

$$\tfrac{34}{3}B = 2 \longrightarrow B = \tfrac{3}{17}, \quad A = -\tfrac{5}{3}\cdot\tfrac{3}{17} = -\tfrac{5}{17}$$

$$Y(t) = -\tfrac{5}{17}\sin t + \tfrac{3}{17}\cos t.$$

General solution is:

$$y = c_1 e^{-t} + c_2 e^{4t} - \tfrac{5}{17}\sin t + \tfrac{3}{17}\cos t.$$

Note that we have also solved

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t$$

Particular solution is:

$$Y(t) = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t.$$

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eg. $y'' - 3y' - 4y = 2e^{4t}$

$Y(t) = Ae^{4t}$ $Y' = 4Ae^{4t}$ $Y'' = 16Ae^{4t}$

$16Ae^{4t} - 12Ae^{4t} - 4Ae^{4t} = 2e^{4t}$

$\underbrace{(16A - 12A - 4A)}_{=0} e^{4t} = 2e^{4t}$

$0 = 2$ oops

What happened? $2e^{4t}$ is a solution to the homogeneous equation.

Try $Y(t) = Ate^{4t}$, $Y' = 4Ate^{4t} + Ae^{4t}$

$Y'' = 16Ate^{4t} + 4Ae^{4t} + 4Ae^{4t}$

$= 16Ate^{4t} + 8Ae^{4t}$

$16\cancel{At}e^{4t} + 8Ae^{4t} - 12\cancel{At}e^{4t} - 3Ae^{4t} - 4\cancel{At}e^{4t} = 2e^{4t}$

$5Ae^{4t} = 2e^{4t}$ $A = \frac{2}{5}$.

$Y(t) = \frac{2}{5}te^{4t}$. General: $y = c_1e^{-t} + c_2e^{4t} + \frac{2}{5}te^{4t}$, ex, #6 $y'' + y = 3\sin 2t + t\cos 2t$

Find general solution.

i) First solve homogeneous equation.

$y'' + y = 0 \longrightarrow r^2 + 1 = 0 \longrightarrow r = \pm i$

$y_1 = \cos(t) + i\sin(t) \quad y_2 = \cos(t) - i\sin(t)$ ok

OR $y_1 = \cos(t) \quad y_2 = \sin(t)$. (preferred)

try $Y(t) = A\sin(2t) + Bt\cos(2t) \left(+ C\cos(2t)\right)$

leave out for now.

$Y'(t) = 2A\cos(2t) - 2Bt\sin(2t) + B\cos(2t)$

$Y''(t) = -4A\sin(2t) - 4Bt\cos(2t) - 2B\sin(2t)$

$-2B\sin(2t)$

$= (-4A - 4B)\sin(2t) - 4Bt\cos(2t)$

$(-4A - 4B)\sin(2t) - 4Bt\cos(2t) + A\sin(2t) + Bt\cos(2t)$

$= 3\sin(2t) + t\cos(2t)$.

$(-3A - 4B)\sin(2t) - 3Bt\cos(2t) = 3\sin(2t) + t\cos(2t)$

$-3A - 4B = 3 \rightarrow -3A + \frac{4}{3} = 3 \rightarrow -3A = \frac{5}{3}$

$-3B = 1 \longrightarrow B = \frac{-1}{3} \quad A = -\frac{5}{9}$

$Y(t) = -\frac{5}{9}\sin(2t) - \frac{1}{3}t\cos(2t)$

General solution:

$$y = c_1 \cos(t) + c_2 \sin(t) - \frac{5}{9}\sin(2t) - \frac{1}{8} t \cos(2t).$$

eg #5 $y'' + 9y = t^2 e^{3t} + 6$

Good guess for $Y(t)$:

$Y(t) = At^2 e^{3t} + B$ (Maybe)

More general guess:

$Y(t) = (At^2 + Bt + C)e^{3t} + D$

$\to Y' = 3At^2 e^{3t} + 2At e^{3t}$

$Y'' = 9At^2 e^{3t} + 12Ate^{3t} + 2Ae^{3t}$

$$9At^2 e^{3t} + 12Ate^{3t} + 2Ae^{3t} + 9At^2 e^{3t} + 9B = t^2 e^{3t} + 6$$

No solution here!