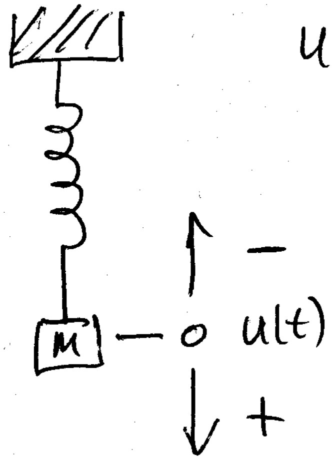
Mechanical + Electrical Vibrations



$u(t)$ = position of mass at time t

$$m u'' + \gamma u' + k u = F$$

m: mass, γ: damping constant, k: spring constant, F: external applied force.

<u>e.g. 1.</u> Need m, γ, k, $F = 0$

4 lbs weight measures force not mass

$F = ma \qquad 4\text{ lb} = m \cdot 32\,\frac{\text{ft}}{\text{s}^2} \qquad m = \frac{1}{8}\,\frac{\text{lb}\cdot\text{s}^2}{\text{ft}}$

$F_{damp} = \gamma \cdot \text{velocity} \qquad 6\text{ lb} = \gamma \cdot 3\,\frac{\text{ft}}{\text{sec}} \qquad \gamma = 2\,\frac{\text{lb}\cdot\text{s}}{\text{ft}}$

$F_{spring} = k \cdot \text{displacement} \qquad 4\text{ lb} = k \cdot \frac{1}{6}\text{ ft} \qquad k = 24\,\frac{\text{lb}}{\text{ft}}$

$$\frac{1}{8}u'' + 2u' + 24u = 0 \qquad u(0) = \frac{1}{2} \qquad u'(0) = 0$$

$u(0)$: initial displacement or position; $u'(0)$: initial velocity

$$u'' + 16u' + 192u = 0$$

Look first at unforced systems, $F=0$.

1. Assume no damping, i.e. $\gamma=0$.

Go back to our e.g.

$$u''+192u=0 \qquad u(0)=\frac{1}{2} \qquad u'(0)=0$$

$r^2+192=0$

$r=\pm 8\sqrt{3}\,i$

$y=c_1\cos(8\sqrt{3}t)+c_2\sin(8\sqrt{3}t)$

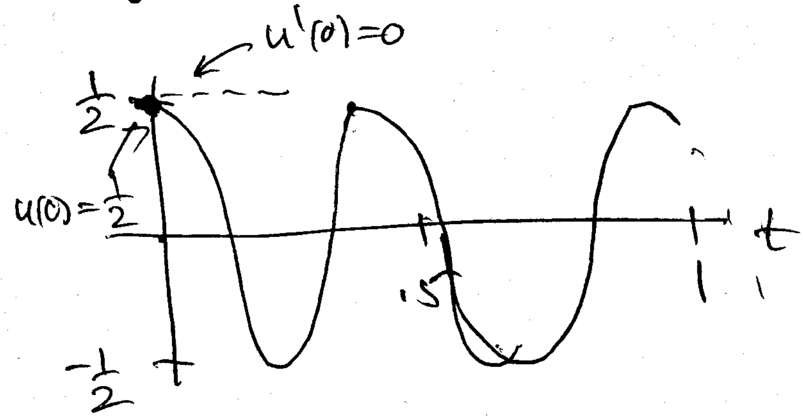
$y'=-8\sqrt{3}c_1\sin(8\sqrt{3}t)$
$\quad +8\sqrt{3}c_2\cos(8\sqrt{3}t)$

$\underline{\frac{1}{2}=c_1} \qquad 0=8\sqrt{3}c_2 \rightarrow \underline{c_2=0}$

$u(t)=\frac{1}{2}\cos(8\sqrt{3}t)$

period: $\frac{2\pi}{8\sqrt{3}} \approx .45$

starts at $(0,\frac{1}{2})$



natural frequency
$=\omega_0=8\sqrt{3}\ \frac{\text{rad}}{\text{sec}}$.
$=\sqrt{\frac{k}{m}}$

amplitude $=\frac{1}{2}\ (=R)$

phase shift $=0\ (=\delta)$

In general.

$$u(t) = \underline{A\cos\omega_0 t + B\sin\omega_0 t}$$
$$= \underline{R\cos(\omega_0 t - \delta)}$$

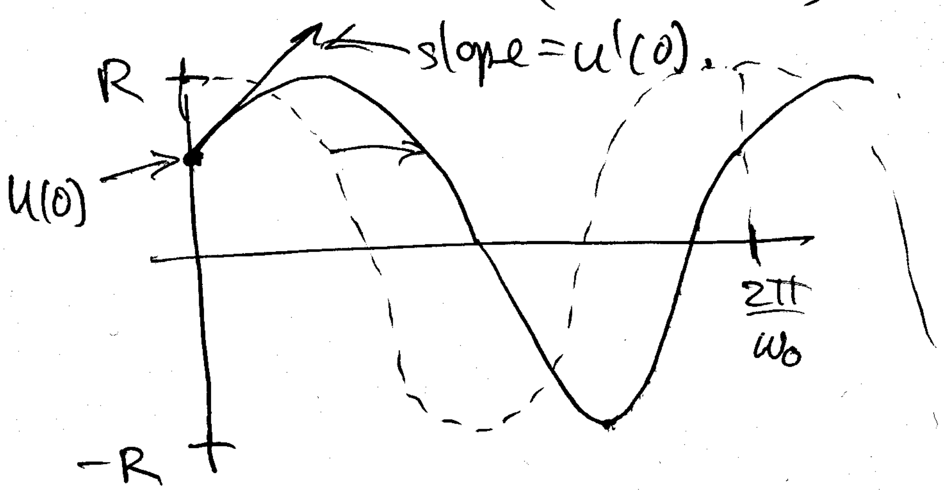
$$R\cos(\omega_0 t - \delta) = R\cos(\omega_0 t)\cos\delta + R\sin(\omega_0 t)\sin\delta$$

$$A = R\cos\delta \qquad \longrightarrow \quad R = (A^2+B^2)^{1/2}$$
$$B = R\sin\delta$$

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = \frac{B}{A}$$

$$mu'' + ku = 0 \qquad u(0) = u_0 \quad u'(0) = u_0'$$

$$u(t) = R\cos(\omega_0 t - \delta) \qquad \omega_0 = \sqrt{\frac{k}{m}}$$



Assume damping.

$$u'' + 16u' + 192u = 0 \qquad u(0) = \frac{1}{2} \quad u'(0) = 0$$

$$r^2 + 16r + 192 = 0 \qquad \frac{2\pi}{8\sqrt{2}} \approx .55$$

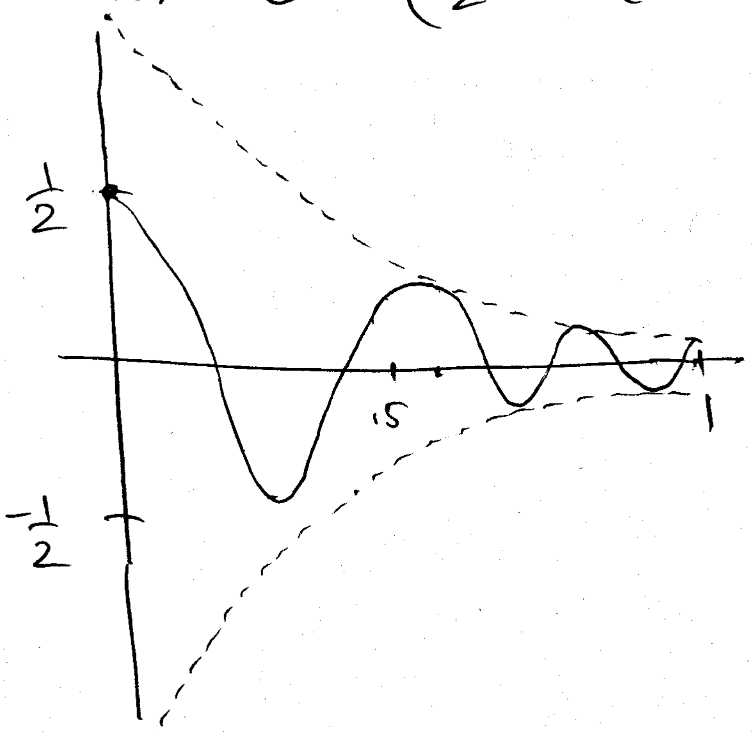
$$r = -8 \pm 8\sqrt{2}\, i$$

$$u(t) = e^{-8t}\left(c_1 \cos(8\sqrt{2}t) + c_2 \sin(8\sqrt{2}t)\right)$$

solving using initial conditions.

$$u(t) = e^{-8t}\left(\frac{1}{2}\cos(8\sqrt{2}t)\right) \qquad \mu = 8\sqrt{2}$$

quasi-frequency.



In general,

$$mu'' + \gamma u' + ku = 0 \qquad u(0) = u_0 \qquad u'(0) = u_0'$$

If $\gamma^2 - 4mk < 0$ (or eqivalently $0 < \gamma < 2\sqrt{mk}$)

solution looks like

$$u(t) = e^{-\gamma/2m\, t} R\cos(\mu t - \delta)$$

$\frac{\gamma}{2m} > 0$

quasi frequency (different from ω_0)

called damped free vibrations.

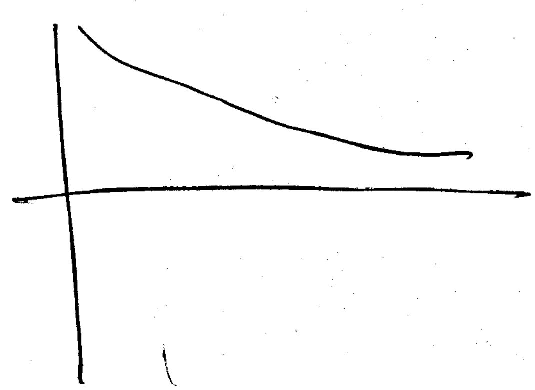
What if $\gamma \geq 2\sqrt{mk}$? This means $\gamma^2 - 4mk \geq 0$.

So $r = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$ Note $\dfrac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m} < 0$

Also $\dfrac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} < 0$

$$u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

both $r_1, r_2 < 0$.

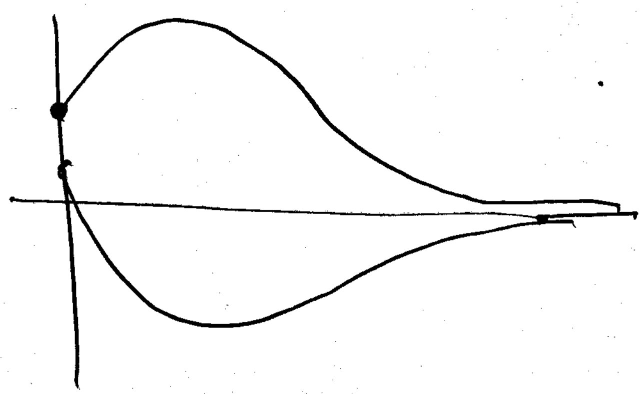


overdamped

$\gamma = 2\sqrt{mk} \longrightarrow \gamma^2 - 4mk = 0$

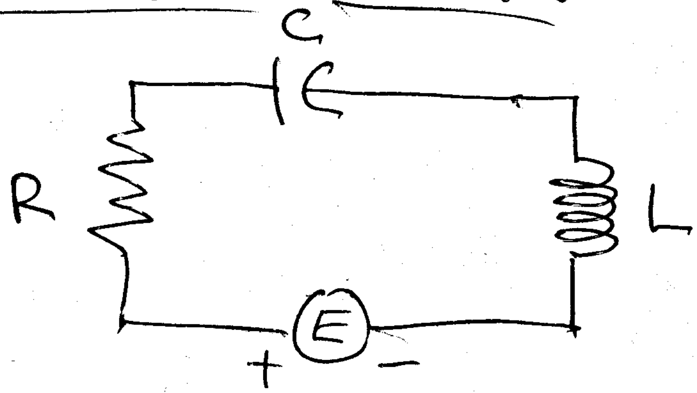
$r = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$ $\quad r_1 = r_2 = \dfrac{-\gamma}{2m}$ double root.

$u(t) = c_1 e^{rt} + c_2 t e^{rt} = (c_1 + c_2 t) e^{-\gamma/2m\, t}$



critically damped

Electrical circuits



$Q(t)$ = charge on capacitor at time t

$LQ'' + RQ' + \frac{1}{C} Q = E$ $\quad Q(0) = Q_0$ ← initial charge

$Q'(0) = Q_0'$ ← initial current.

↑ inductance ↑ resistance ↑ capacitance

Looking at $E=0$ (no external voltage)

$R=0$ (no resistance / no damping)

$$Q(t) = A\cos(\omega_0 t - \delta)$$

This gives an oscillating current in circuit

We can also interpret damped oscillations

(i.e. $R \neq 0$, R small gives damped oscillations

R large, no oscillations)

3.8 Forced vibrations

Here $mu'' + \gamma u' + ku = F \quad \underline{\underline{F \neq 0}}$

1. Assume no damping (Forced Free Vibrations)

Assume periodic forcing $F(t) = F_0 \cos(\omega t)$

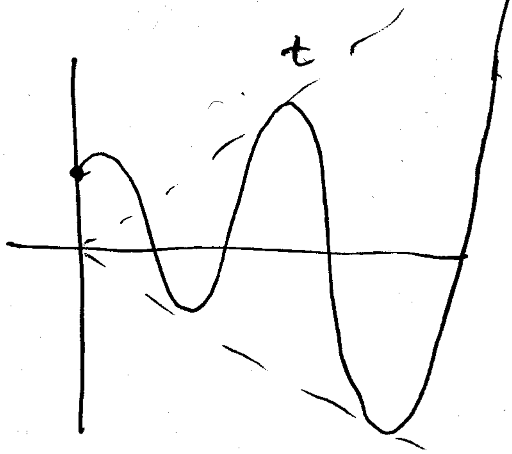
$$mu'' + ku = F_0 \cos(\omega t)$$

$$u(t) = \underbrace{c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)}_{R\cos(\omega_0 t - \delta)} + \underline{Y(t)}$$

ω_0 = natural frequency of system.

A. If $\omega = \omega_0$ then using undetermined coeffs we look for $Y(t) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$

So $u(t) = R_1 \cos(\omega_0 t - \delta_1) + \underline{\underline{t}}\, R_2 \cos(\omega_0 t - \delta)$



in this case $u(t) \to \infty$ as $t \to \infty$

called <u>resonance</u>.

B. If $\omega \neq \omega_0$ $\quad Y(t) = A\cos(\omega t) + B\sin(\omega t)$

e.g. $u'' + u = \cos 2t$

$r^2 + 1 = 0$ $\quad Y(t) = A\cos 2t + B\sin 2t$

$r = \pm i$ $\quad Y'(t) = -2A\sin 2t + 2B\cos 2t$

$Y''(t) = -4A\cos 2t - 4B\sin 2t$

$u(t) = c_1\cos t + c_2\sin t + A\cos 2t + B\sin 2t$

$-4A\cos 2t - 4B\sin 2t + A\cos 2t + B\sin 2t = \cos 2t$

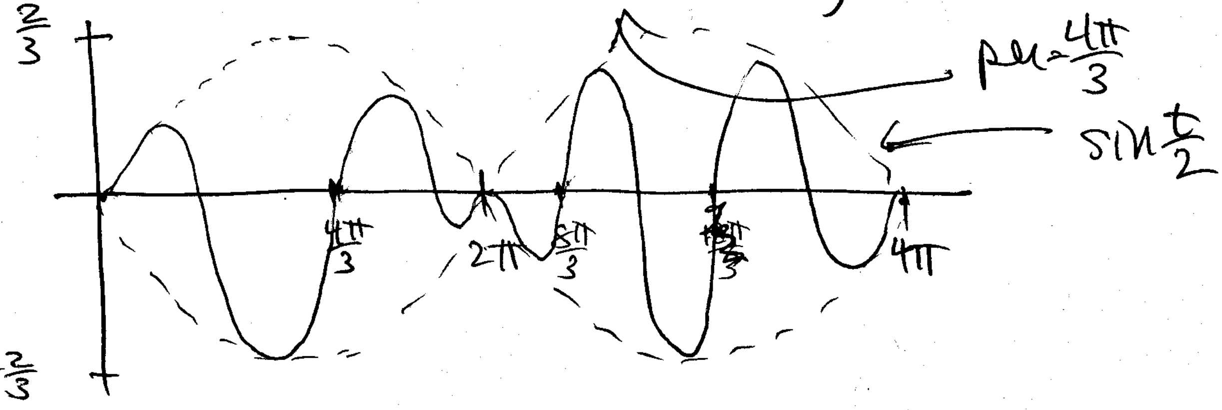
$-3B = 0 \quad -3A = 1 \qquad B = 0 \quad A = -\frac{1}{3}$

$u(t) = c_1\cos t + c_2\sin t - \frac{1}{3}\cos 2t$

Say $u(0) = 0$, $u'(0) = 0$.

$0 = c_1 - \frac{1}{3}$; $0 = c_2$ $\qquad c_1 = \frac{1}{3}$ $\quad c_2 = 0$

$$u(t) = \frac{1}{3}(\cos t - \cos 2t)$$
$$= \frac{1}{3}\left(-2\left(\sin\frac{3t}{2}\right)\left(\sin\left(-\frac{t}{2}\right)\right)\right)$$
$$= \frac{2}{3}\sin\left(\frac{3t}{2}\right)\sin\left(\frac{t}{2}\right) \leftarrow \text{per} = 4\pi$$



In general.

$$mu'' + ku = F_0 \cos \omega t \qquad \omega \neq \omega_0$$

$$u(t) = \underbrace{c_1 \cos \omega_0 t + c_2 \sin \omega_0 t}_{\text{initial conditions}} + \underbrace{A \cos \omega t + B \sin \omega t}_{\text{undet coeffs gives}}$$

$u(0)=0 \quad u'(0)=0$ gives

$$c_1 = \frac{-F_0}{m(\omega_0^2 - \omega^2)}, \quad c_2 = 0 \qquad A = \frac{F_0}{m(\omega_0^2-\omega^2)}, \quad B = 0$$

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t)$$

$$= \frac{F_0}{m(\omega_0^2 - \omega^2)} \underbrace{\sin\left(\frac{\omega_0 - \omega}{2} t\right)}_{\text{slow oscillation}} \underbrace{\sin\left(\frac{\omega_0 + \omega}{2} t\right)}_{\text{fast oscillation}}$$

This is called "beat".