Looked at $\frac{dy}{dt} = f(t,y)$

$\frac{dy}{dt} = Ay + B$, A, B constants.

Falling object

$$\frac{dv}{dt} = g - \frac{\gamma}{m} v$$

Mice + Owls

$$\frac{dp}{dt} = rp - k$$

B. $\frac{dv}{dt} = 9.8 - .2v = \frac{49 - v}{5}$

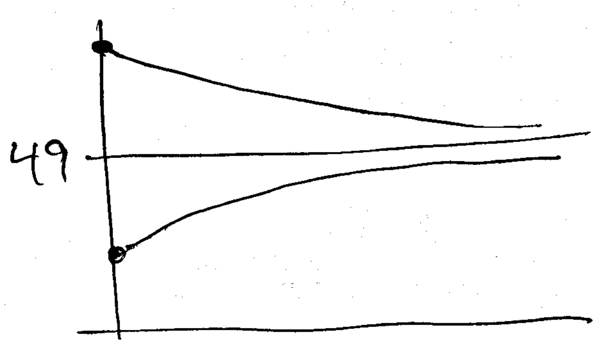
$$\frac{dv}{49 - v} = \frac{1}{5} dt$$

$$v = 49 - \underline{\underline{c}}\, e^{-1/5 t}$$

$v(0) = 49 - c$

$\underline{c = 49 - v(0)}$

v



t

$v(0) > 49$

$c = 49 - v(0) < 0$

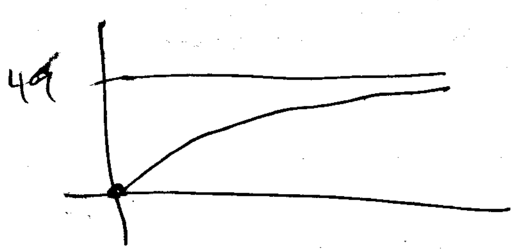
$v(0) < 49$

$c > 0$

<u>e.g.</u> $v(0) = 0 \longrightarrow$ object is dropped.

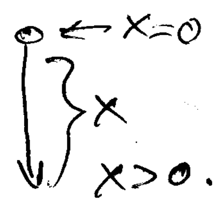
$c = 49 \qquad v(t) = 49 - 49 e^{-1/5 t}$

$= 49(1 - e^{-1/5 t})$



If object is dropped from 300 m when does it hit the ground?

$v(t) = \frac{dx}{dt}$ $\quad x =$ distance the object has fallen.



$$\frac{dx}{dt} = 49(1 - e^{-1/5 t})$$

$$x(t) = 49(t + 5e^{-1/5 t}) + c \quad \text{Find } c.$$

$$x(0) = 0 \qquad 0 = 49(5) + c = 245 + c$$

$$c = -245$$

$$x(t) = 49(t + 5e^{-1/5 t}) - 245$$

Solve: $49(t + 5e^{-1/5 t}) - 245 = 300$

$$49(t + 5e^{-1/5 t}) = 545$$

$$t + 5e^{-1/5 t} = \frac{545}{49} \approx 11.1 \leftarrow \text{solve } \underline{\text{numerically}}$$

$$t \approx 10.5 \text{ sec.}$$

How fast at impact?

$$v(10.5) = 49(1 - e^{-1/5(10.5)}) \approx 43.0 \text{ m/sec}$$

#2) (a) $\frac{dy}{dt} = y - 5 \quad y(0) = y_0$

$$\frac{dy}{y-5} = dt \longrightarrow \ln|y-5| = t + c$$

$$|y-5| = e^{c} e^{t}$$

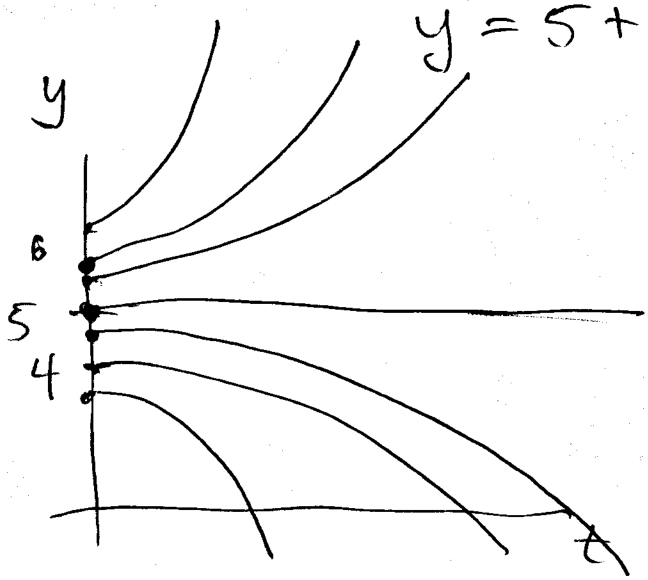
$$y - 5 = c e^{t}$$

$$y = 5 + c e^{t}$$

$$y(0) = 5 + c = y_0$$

$$c = y_0 - 5$$

$$y = 5 + (y_0 - 5) e^{t}$$



$y_0 = 4$

$y = 5 - e^{t}$

$y_0 = 6$

$y = 5 + e^{t}$

#(9) (a) q = amount of dye in pool in kg.

t = time in minutes.

Concentration of dye: $\frac{q(t)}{60000}\ \frac{kg}{gal}$

How much dye is removed by filter?

$$200\ \frac{gal}{min} \cdot \frac{q(t)}{60000}\ \frac{kg}{gal} = \frac{q(t)}{300}\ \frac{kg}{min}.$$

$$\frac{dq}{dt} = -\frac{q}{300} \qquad q(0)=5 \quad //$$

(b) $q(t) = c\,e^{-t/300}$ $\qquad q(0)=5$

$q(0)=c \qquad q(t)=5e^{-t/300}$ //

(c) We need to have $\frac{q}{60000} < .00002$

or $q < 1.2$ kg. $\quad 4h = 240$ m.

$$q(240) = 5e^{-\frac{240}{300}} = 5e^{-.8} \approx 2.25$$

(d) Solve $q(T) = 1.2$ i.e. $5e^{-T/300} = 1.2$

(e) IVP becomes $\frac{dq}{dt} = -\frac{F}{60000}q$; $q(0)=5$

1.3 Classification of DEs.

A. ODE vs. PDE

ordinary   partial

Ordinary DE involves $y, y', y'', \ldots$
so $y = y(t)$ (one variable)

Partial DE involves $\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}, \frac{\partial y}{\partial z}, \ldots$

so $y = y(t, x, z, w, \ldots)$

B. Systems of ODE.

$$\frac{dx}{dt} = f(t, x, y)$$
$$\frac{dy}{dt} = g(t, x, y)$$

solution is two functions $x(t), y(t)$.

e.g.

$$\frac{dx}{dt} = ax + by + ct$$
$$\frac{dy}{dt} = dx + ey + ft$$

a, b, c, d, e, f constants.

C. Order of an ODE

order = highest derivative appearing in the ODE or system of ODEs.

e.g. $y''' + (y')^2 + e^t y = 0$ order = 3

In general an n^{th} order ODE looks like

$$F(t, y, y', y'', \dots, y^{(n)}) = 0$$

implicit form of ODE

some function of $n+2$ variables

Above example corresponds to:

$$F(t, y, y', y'', y''') = 0$$

$$F(x_1, x_2, x_3, x_4, x_5) = x_5 + x_3^2 + e^{x_1} x_2 .$$

In this class we will assume we can write.

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$$

explicit form of ODE.

e.g. $y''' = -(y')^2 - e^t y$.

D. ~~Linear~~ Linear vs. Non-linear.

A linear n^{th} order ODE looks like

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t)$$

Non-linear means you can't do this.

eg $y''' + (y')^2 + e^t y = 0$ 3rd order non-linear

$y''' + \underbrace{2\tan(t)}\, y' + \underbrace{e^t} y = \underbrace{4t^2}$ 3rd order linear.

$\cancel{\frac{d^2\theta}{dt^2} + \frac{g}{L}}$ $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$ 2nd order non-linear.

$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$ ($\theta = y$ here)

2nd order linear.

$\underline{y}\, y' + 2ty + \sec(t) = 0$ 1st order non-linear

$$\boxed{\frac{dy}{dt} = f(t,y) \quad \text{1st order (explicit form) (maybe linear, maybe not)}}$$

#15) $y''' - 3y'' + 2y' = 0$

3rd order linear (constant coefficients, homogeneous) equations like this have solutions of the form $y = e^{rt}$ for certain values of r.

If $y = e^{rt}$

$y' = re^{rt}$

$y'' = r^2 e^{rt}$

$y''' = r^3 e^{rt}$

$r^3 e^{rt} - 3r^2 e^{rt} + 2re^{rt} = 0$

$e^{rt}(r^3 - 3r^2 + 2r) = 0$

$r^3 - 3r^2 + 2r = 0$

$r(r^2 - 3r + 2) = 0$

$r(r-1)(r-2) = 0$

$r = 0,\ r = 1,\ r = 2$ //.

2.1 Linear equations; Integrating factors

$$\frac{dy}{dt} = f(t,y) \leftarrow 1^{st} \text{ order.}$$

Our examples: $\frac{dv}{dt} = g - \frac{\gamma}{m} v$ Falling object

$\frac{dp}{dt} = rp - k$ Mice/owls.

General form: $\frac{dy}{dt} + ay = b.$ a, b const.

We can use separation of variables.

$$\frac{dy}{dt} = -ay + b = -a\left(y - \frac{b}{a}\right)$$

$$\frac{dy}{y - \frac{b}{a}} = -a\,dt \rightarrow \ln\left|y - \frac{b}{a}\right| = e^{-at} + c$$

$$\rightarrow \left|y - \frac{b}{a}\right| = e^{c} e^{-at}$$

$$\rightarrow y - \frac{b}{a} = c\,e^{-at} \rightarrow \underline{\underline{y = \frac{b}{a} + c\,e^{-at}}}$$

Different method.

<u>Idea</u>: Integrating factor.

Suppose we replace y by $\mu(t) y$

Then $\frac{d}{dt}(\mu \cdot y) = \mu(t) \frac{dy}{dt} + \mu'(t) y$

This looks like LHS of $\frac{dy}{dt} + ay = b$.

If $\underline{\mu'(t) = a\mu(t)}$ then I would have

$$\frac{d}{dt}(\mu \cdot y) = \mu(t)\frac{dy}{dt} + a\mu(t)\,y$$
$$= \mu(t)\left(\frac{dy}{dt} + ay\right).$$

This would mean my equation becomes

$$\frac{d}{dt}(\mu \cdot y) = \mu(t)\left(\frac{dy}{dt} + ay\right) = \mu(t)\,b.$$

If $\mu'(t) = a\mu(t)$ then $\mu(t) = e^{at}$

Solve:
$$\frac{dy}{dt} + ay = b$$
$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}b$$
$$\frac{d}{dt}(e^{at}y) = e^{at}b.$$
$$e^{at}y = \frac{b}{a}e^{at} + c$$
$$y = \frac{b}{a} + ce^{-at}$$

Same solution as before.