Quiz 10 Section 4.2

Exam 2 - Tues 11/8 Coverage on website

Laplace Transforms $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

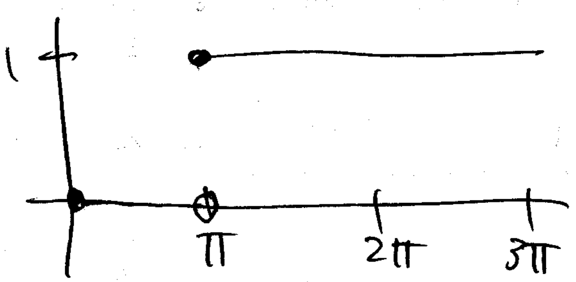
Laplace transforms of step functions or piecewise functions.

$$u_c(t) = \begin{cases} 1 & t \geq c \\ 0 & t < c \end{cases}$$

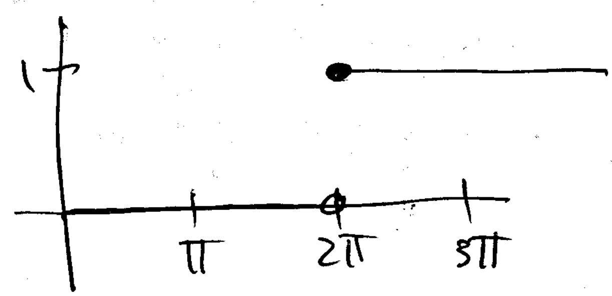
$(c > 0)$

~~$\mathcal{L}\{u_c(t)\} =$~~

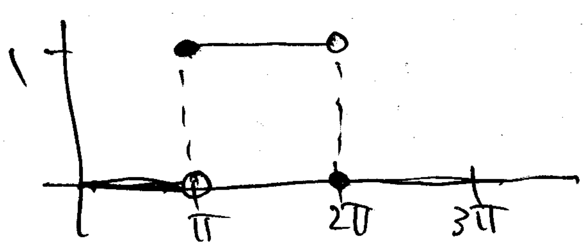
e.g. $h(t) = u_\pi(t) - u_{2\pi}(t)$



$u_\pi(t)$



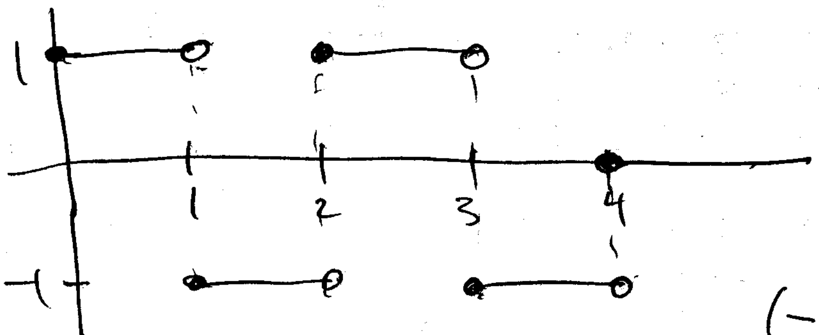
$u_{2\pi}(t)$



rectangular pulse.

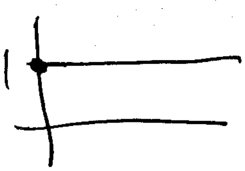
eg #8)

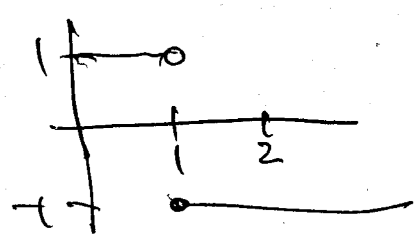
$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ -1 & 1 \le t < 2 \\ 1 & 2 \le t < 3 \\ -1 & 3 \le t < 4 \\ 0 & t \ge 4 \end{cases}$$

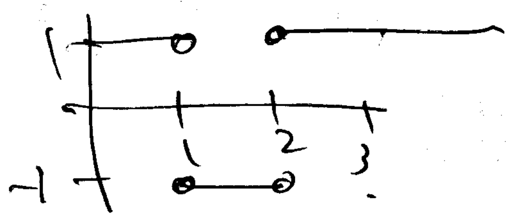


$(-1-1) = (\text{new} - \text{old})$

$f_1(t) = 1 \qquad f_2(t) = f_1(t) - 2u_1(t) \qquad f_3(t) = f_2(t) + 2u_2(t)$







$f_4(t) = f_3(t) - 2u_3(t) \qquad f_5(t) = f_4(t) + u_4(t) = f(t)$

$f(t) = 1 - 2u_1(t) + 2u_2(t) - 2u_3(t) + u_4(t)$.

Laplace transform of $u_c(t)$.

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt = \int_c^\infty e^{-st}\,dt = \frac{-1}{s} e^{-st}\Big|_c^\infty$$

$$= \frac{1}{s} e^{-cs}, \quad s > 0.$$

e.g. What about $\mathcal{L}\{f(t)\}$, f our first example?

$$f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t)$$

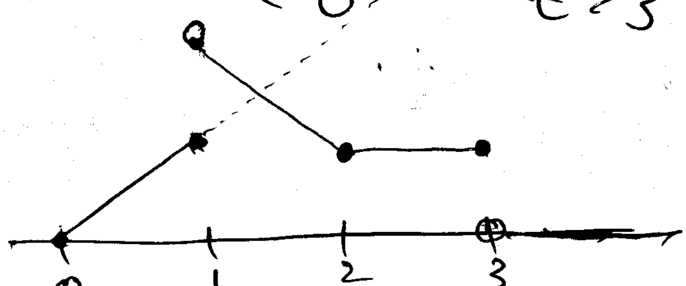
$$\mathcal{L}\{f(t)\} = \mathcal{L}\{2\} + 3\mathcal{L}\{u_4\} - 6\mathcal{L}\{u_7\} + 2\mathcal{L}\{u_9\}$$

$$= \frac{2}{s} + \frac{3}{s}e^{-4s} - \frac{6}{s}e^{-7s} + \frac{2}{s}e^{-9s}$$

$$= \frac{1}{s}\left(2 + 3e^{-4s} - 6e^{-7s} + 2e^{-9s}\right)$$

e.g.

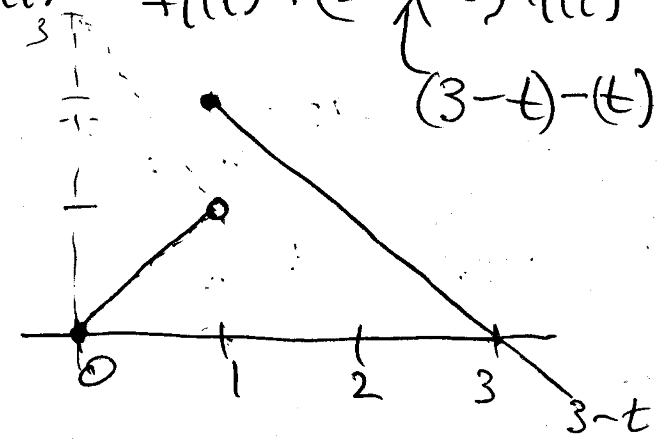
$$f(t) = \begin{cases} t & 0 \le t \le 1 \\ 3-t & 1 < t \le 2 \\ 1 & 2 < t \le 3 \\ 0 & t > 3 \end{cases}$$



Write f in terms of Heaviside functions

$f_1(t) = t$

$f_2(t) = f_1(t) + (3-\underbrace{2t}_{(3-t)-(t)})u_1(t)$



$\boxed{t = 3/2 \quad f(t) = 3/2}$

$\boxed{f_2(3/2) = \frac{3}{2} + (3 - 2\cdot\frac{3}{2}) = \frac{3}{2}}$

$t = 1 \quad f(1) = 1$

$f_2(1) = 1 + (3-2)(1) = 2$

Why different?
Does not matter for
Laplace transforms

$f_3(t) = f_2(t) + (1-(3-t))u_2(t)$

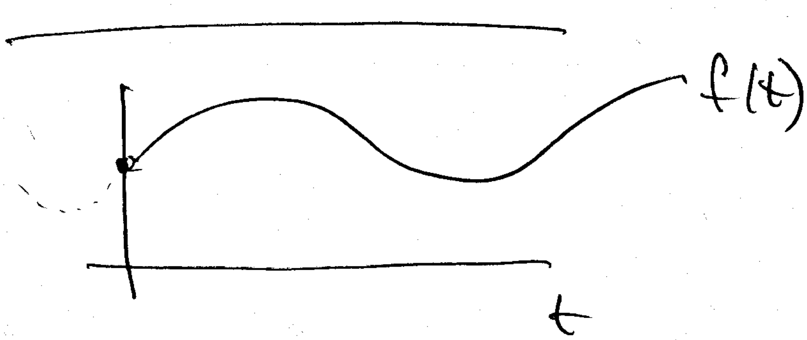
$\quad = f_2(t) + (t-2)\, u_2(t).$

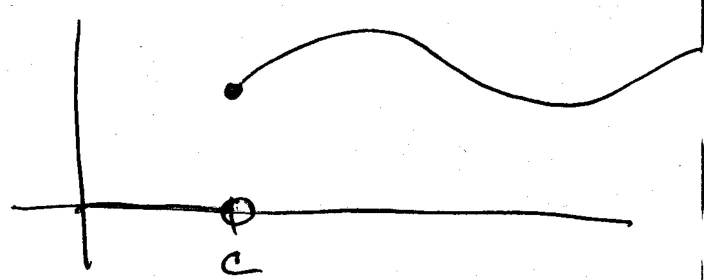
$f_4(t) = f_3(t) + (0-1)u_3(t) = f_3(t) - u_3(t) = f(t)$

$f(t) = t + (3-2t)u_1(t) + (t-2)u_2(t) - u_3(t).$

How do we compute $\mathcal{L}\{f(t)\}$?

$\cancel{\mathcal{L}\{(3-2t)u_1(t)\} = (3-2t)\mathcal{L}\{u_1(t)\}}$
WRONG!





$u_c(t) f(t-c)$

$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty u_c(t) f(t-c) e^{-st}\,dt$$

$$= \int_c^\infty f(t-c) e^{-st}\,dt \qquad \begin{aligned} u &= t-c \\ du &= dt \\ t &= u+c \end{aligned}$$

$$= \int_0^\infty f(u) e^{-s(u+c)}\,du \qquad \begin{aligned} t=c &\to u=0 \\ t=\infty &\to u=\infty \end{aligned}$$

$$= e^{-sc} \underbrace{\int_0^\infty f(u) e^{-su}\,du}_{\mathcal{L}\{f(t)\}}$$

$$= e^{-sc}\mathcal{L}\{f(t)\}.$$

$$\boxed{\mathcal{L}\{u_c(t)f(t-c)\} = e^{-sc}\mathcal{L}\{f(t)\}}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} + \mathcal{L}\{(3-2t)u_1\} + \mathcal{L}\{\underbrace{(t-2)}_{f(t-2)}u_2\} - \mathcal{L}\{u_3\}$$

already in right form. $f(x)=x$

$$(3-2t)u_1(t)$$

$$= (3-2(t-1)-2)u_1(t)$$

$$= \underbrace{(1-2(t-1))}_{f(t-1)}u_1(t)$$

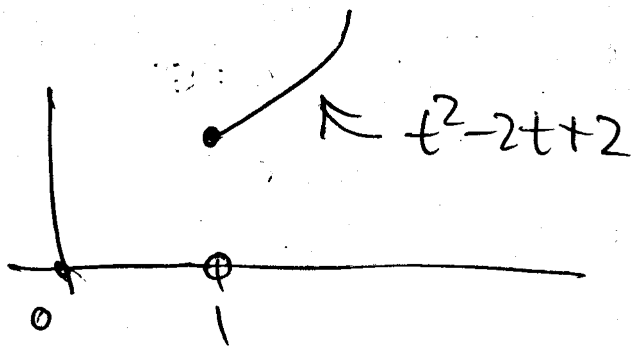
$$f(x) = 1-2x$$

$$= \mathcal{L}\{t\} + e^{-s}\mathcal{L}\{1-2t\} + e^{-2s}\mathcal{L}\{t\} - \mathcal{L}\{u_3\}$$

$$= \frac{1}{s^2} + e^{-s}\left(\frac{1}{s} - \frac{2}{s^2}\right) + e^{-2s}\frac{1}{s^2} - \frac{e^{-3s}}{s}$$

$$= \frac{1}{s^2}\left(1 - 2e^{-s} + e^{-2s}\right) + \frac{1}{s}\left(e^{-s} - e^{-3s}\right)$$

eg #14)
$$f(t) = \begin{cases} 0 & t < 1 \\ t^2 - 2t + 2 & t \geq 1 \end{cases} = (t^2 - 2t + 2)\,u_1(t)$$



need to write this as $g(t-1)$ for some g.

$(t^2 - 2t + 1) + 2 - 1$

$(t-1)^2 + 1$

$= g(t-1)\,u_1(t)$ where $g(x) = x^2 + 1$

$g(t-1) = (t-1)^2 + 1$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t-1)\,u_1\} = e^{-s}\,\mathcal{L}\{g(t)\} = e^{-s}\,\mathcal{L}\{t^2+1\}$$
$$= e^{-s}\left(\frac{2}{s^3} + \frac{1}{s}\right)$$

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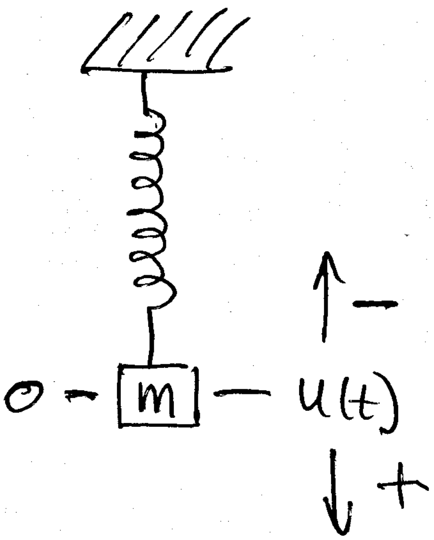
eg $F(s) = \dfrac{3!}{(s-2)^4}$ Find $\mathcal{L}^{-1}\{F(s)\}$.

$$\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = t^3$$
$$\mathcal{L}^{-1}\left\{\frac{3!}{(s-2)^4}\right\} = e^{2t}t^3$$

3.7 Mechanical/Electrical vibrations

Mass + Spring systems.



$u(t)$ = position of mass at time t.

$$mu'' + \gamma u' + ku = F$$

m = mass

k = spring constant

Hooke's Law:
$F_{spring} = k$ (displacement)

γ = damping constant

$F_{damp} = \gamma$ (velocity)
like air resistance

F = forcing term, i.e., external force applied to the system.

Note: $m, k > 0$, $\gamma \geq 0$