Final Exam 12/13 $1^{30}-4^{15}$

Information on the final is on web site.

Quiz 14 12/8 7.1, 7.3

Goal: Solve $\vec{x}' = A\vec{x}$ A const $n \times n$ matrix

Eigenvalues/Eigenvectors

A : $n \times n$ matrix

λ is an eigenvalue of A if $A\vec{x} = \lambda\vec{x}$ for some non-zero vector $\vec{x}$. In this case $\vec{x}$ is an eigenvector corresponding to λ.

How do you find eigenvalues/vectors?

$A\vec{x} = \lambda\vec{x} \iff A\vec{x} - \lambda\vec{x} = \vec{0} \iff (A - \lambda I)\vec{x} = \vec{0}$

In order for this to have non-zero solution we must have $\boxed{\det(A - \lambda I) = 0}$

e.g. $A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$ Find eigenvalues

$$A - \lambda I = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda)+4$$

$$= \lambda^2 - \lambda - 2$$

$$\lambda^2 - \lambda - 2 = 0 \rightarrow (\lambda+1)(\lambda-2) = 0$$

$$\boxed{\lambda = -1,\ \lambda = 2}$$

<u>e.g.</u> ~~#~~ $\boxed{\#22\ p\,373}$

$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ Recall we found eigenvectors $\vec{\xi}^{(1)} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ $\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

What are the eigenvalues?

$$A\vec{\xi}^{(1)} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad \lambda = 2$$

$$A\vec{\xi}^{(2)} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \lambda = -1$$

$$(A-2I) = \begin{pmatrix} 3-2 & -2 \\ 2 & -2-2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \quad \det = -4+4 = \underline{0}$$

$$(A-(-1)I) = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}_2 \quad \det = -4+4 = \underline{0}$$

How do you find eigenvectors?

Solve $(A-\lambda I)\vec{x}=\vec{0}$ if λ is an eigenvalue.

e.g. $A=\begin{pmatrix}3 & -1\\ 4 & -2\end{pmatrix}$ $\quad \lambda=2$, $\lambda=-1$

$\lambda=2$: Solve $\begin{pmatrix}3-2 & -1\\ 4 & -2-2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

$$\begin{pmatrix}1 & -1\\ 4 & -4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}1 & -1 & 0\\ 4 & -4 & 0\end{pmatrix}\xrightarrow{-4}\begin{pmatrix}1 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\quad \begin{matrix}x_1-x_2=0\\ x_1=t\\ x_2=t\end{matrix}$$

$$\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}t\\ t\end{pmatrix}=t\underline{\underline{\begin{pmatrix}1\\ 1\end{pmatrix}}},\ t\in\mathbb{R}.$$

$\lambda=-1$ Solve $\begin{pmatrix}4 & -1\\ 4 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

$4x_1-x_2=0$ $\quad \begin{matrix}x_1=\frac{1}{4}t\\ x_2=t\end{matrix}$ $\quad \begin{pmatrix}x_1\\ x_2\end{pmatrix}=t\underline{\underline{\begin{pmatrix}\frac{1}{4}\\ 1\end{pmatrix}}}\ t\in\mathbb{R}.$

or could say $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $s \in \mathbb{R}$.

eg #25) $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)\left[(-\lambda)(3-\lambda)-4\right] - 2\left[2(3-\lambda)-8\right]$$

$$+4\left[4+4\lambda\right]$$

$$= (3-\lambda)\left[\lambda(\lambda-3)-4\right] - 4\left[3-\lambda-4-4-4\lambda\right]$$

$$= (3-\lambda)\left[\lambda^2-3\lambda-4\right] - 4\left[-5\lambda-5\right]$$

$$= (3-\lambda)(\lambda+1)(\lambda-4) + (4)(5)(\lambda+1)$$

$$= (\lambda+1)\left((3-\lambda)(\lambda-4)+20\right)$$

$$= (\lambda+1)(-\lambda^2+7\lambda+8)$$

$$= -(\lambda+1)(\lambda^2-7\lambda-8) = -(\lambda+1)(\lambda+1)(\lambda-8)$$

Eigenvalues $\lambda=-1$, $\lambda=8$

Note: A is 3×3 but we only get 2 eigenvalues

<u>$\lambda=-1$:</u> Solve $(A-\lambda I)\vec{x}=\vec{0}$

$$\left(\begin{array}{cccc} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right) \xrightarrow{G.E.} \left(\begin{array}{cccc} 4 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

<u>or</u> reduces to one equation: $2x_1+x_2+2x_3=0$

$$2x_1=-x_2-2x_3$$
$$x_1=-\tfrac{1}{2}x_2-x_3$$

$$x_1=-\tfrac{1}{2}s-t$$
$$x_2=s$$
$$x_3=t$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Two linearly independent eigenvectors for $\lambda=-1$; e.g. $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

<u>$\lambda=8$:</u> $\frac{4}{5}$ $\left(\begin{array}{cccc} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc} -5 & 2 & 4 & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \\ 0 & \frac{18}{5} & -\frac{9}{5} & 0 \end{array}\right)$

$$\rightarrow \left(\begin{array}{cccc} -5 & 2 & 4 & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1=\tfrac{3}{5}t$$
$$x_2=-\tfrac{1}{2}t$$
$$x_3=t$$

$$-2x_2+x_3=0$$
$$-5x_1+2x_2+4x_3=0$$
$$-5x_1-t+4t=0$$
$$-5x_1+3t=0$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 3/5 \\ -1/2 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

7.4 Basic Theory of 1st order linear systems.

Recall basic theory of nth order linear homogeneous equations.

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0$$

1. Try to find n solutions, $y_1, y_2, y_3, \ldots, y_n$ in some way. Are they a fundamental set of solutions? Compute Wronskian!

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

determinant

$\neq 0$ for all t?

2. Any solution can be written uniquely as

$$\varphi(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

3. If the coefficients $p_1, p_2, \cdots, p_n$ are constant then we can always find n solutions by solving characteristic equation. They will automatically be a fundamental set.

1^{st} order linear systems.

$$\vec{x}'(t) = P(t)\vec{x}(t) + \cancel{\vec{g}(t)}$$

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ \vdots & \vdots & & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{pmatrix}$$

$$\vec{g}(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}$$

$\vec{g}(t) = 0$ means system is homogeneous.

1. Want to find n solutions.

e.g. $\vec{x}'(t) = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x}(t)$

$$\rightarrow \vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} e^{2t}$$

Would write

$$\vec{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} = \begin{pmatrix} e^{-t} \\ 2e^{-t} \end{pmatrix}$$

$$\vec{x}^{(2)}(t) = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} e^{2t} = \begin{pmatrix} e^{2t} \\ \frac{1}{2}e^{2t} \end{pmatrix}$$

Check Wronskian:

$$\begin{vmatrix} e^{-t} & e^{2t} \\ 2e^{-t} & \frac{1}{2}e^{2t} \end{vmatrix} = \frac{1}{2}e^{t} - 2e^{t} = -\frac{3}{2}e^{t} \neq 0.$$

In general: Would get

$$\vec{x}^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix} \quad \vec{x}^{(2)}(t) = \begin{pmatrix} x_{12}(t) \\ \vdots \\ x_{n2}(t) \end{pmatrix}, \ldots, \vec{x}^{(n)}(t) = \begin{pmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}$$

To check linear independence, I form

$$X(t) = \begin{pmatrix} | & | & & | \\ \vec{x}^{(1)} & \vec{x}^{(2)} & \cdots & \vec{x}^{(n)} \\ | & | & & | \end{pmatrix}$$

$\{\vec{x}^{(1)}, \dots, \vec{x}^{(n)}\}$ form a fundamental set

if $\det \mathbb{X}(t) \neq 0$ for all t on which solution is defined.

Define $W[\vec{x}^{(1)}, \dots, \vec{x}^{(n)}] = \det \mathbb{X}(t)$.

In fact, W is either identically zero or never zero.

2. Any solution to $\vec{x}' = P(t)\vec{x}$ can be written uniquely as

$$\vec{\varphi}(t) = c_1 \vec{x}^{(1)}(t) + \cdots + c_n \vec{x}^{(n)}(t).$$

vector $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ depends on initial cond.

3. We will give a method to always find a fund. set in the case of constant coefficients.