# MATH 213 - EXAM 2 - VERSION 1

1. $\vec{r}(t) = \langle \sin(2t), \cos(2t), 2t \rangle,\ t \geq 0$

$\vec{r}\,'(t) = \langle 2\cos(2t), -2\sin(2t), 2 \rangle$

$|\vec{r}\,'(t)| = (4\cos^2(2t) + 4\sin^2(2t) + 4)^{1/2}$

$= (8)^{1/2} = 2\sqrt{2}.$

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \left\langle \dfrac{\cos 2t}{\sqrt{2}}, \dfrac{-\sin 2t}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} \right\rangle$ //

$\vec{T}\,'(t) = \left\langle \dfrac{-2\sin 2t}{\sqrt{2}}, \dfrac{-2\cos 2t}{\sqrt{2}}, 0 \right\rangle$

$|\vec{T}\,'(t)| = (2\sin^2 2t + 2\cos^2 2t + 0)^{1/2} = \sqrt{2}$

$K(t) = \dfrac{\sqrt{2}}{2\sqrt{2}} = \dfrac{1}{2}$ //

2. $f(x,y,z) = y^2z^2 + x^3y + \frac{xy}{z}$

$f_{yzy} = f_{yyz}$

$f_y = 2yz^2 + x^3 + \frac{x}{z}$

$f_{yy} = 2z^2$

$f_{yyz} = 4z$ //

3. $f(x,y) = x^2y^2 - 2x^3y + 2x$

(a) $\nabla f = \langle 2xy^2 - 6x^2y + 2,\ 2x^2y - 2x^3 \rangle$

(b) $\nabla f(1,2) = \langle 8 - 12 + 2,\ 4 - 2 \rangle = \langle -2, 2 \rangle$

$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1, 3 \rangle}{\sqrt{10}} = \langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle$

$D_{\vec{u}} f(1,2) = \langle -2, 2 \rangle \cdot \langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle = \frac{4}{\sqrt{10}}$ //

(c) Max rate of change: $|\nabla f(1,2)| = \sqrt{8}$ //

Direction of max rate: $\frac{\nabla f(1,2)}{|\nabla f(1,2)|} = \frac{\langle -2, 2 \rangle}{\sqrt{8}} = \langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

(d) $L(x,y) = \underline{f(1,2)} + f_x(1,2)(x-1) + f_y(1,2)(y-2)$

$$\boxed{f(1,2) = 4 - 4 + 2 = \underline{2}}$$

$= 2 - 2(x-1) + 2(y-2)$

$= 2 - 2x + 2 + 2y - 4$

$= -2x + 2y$ //

(e) $df = f_x\,dx + f_y\,dy$

$= -2(.1) + 2(.3) = -.2 + .6 = +.4$

4. $f(x,y) = x^4 + 2y^2 - 4xy$

$f_x = 4x^3 - 4y \quad f_y = 4y - 4x$

$4x^3 - 4y = 0 \longleftrightarrow 4x^3 - 4x = 0$

$4y - 4x = 0 \longrightarrow x = y \quad 4x(x^2 - 1) = 0$

$x = 0 \quad x = 1 \quad x = -1$

$\longrightarrow y = 0 \quad y = 1 \quad y = -1$

critical pts:

$(0,0)\ (1,1)\ (-1,-1)$

$f_{xx} = 12x^2 \quad f_{yy} = 4 \quad f_{xy} = -4$

$D(x,y) = 48x^2 - 16$

$D(0,0) = -16 < 0$

$(0,0)$ is a saddle point

$D(1,1) = 32 > 0$

$f_{xx}(1,1) = 12 > 0$

$(1,1)$ is a local minimum

$D(-1,-1) = +64 \cancel{>0}$

$f_{xx}(-1,-1) > 0$

$(-1,-1)$ is a ~~saddle point~~ local minimum

5. $f(x,y) = y^2 - 4x^2$ $\quad g(x,y) = x^2 + 2y^2 - 4$

$\nabla f = \langle -8x, 2y \rangle$ $\quad \nabla g = \langle 2x, 4y \rangle$

$\nabla f = \lambda \nabla g \longrightarrow -8x = 2\lambda x$

$2y = 4\lambda y.$

$-8x = 2\lambda x \longrightarrow \underline{x=0} \quad \underline{\lambda = -4}$

$x^2 + 2y^2 = 4 \longrightarrow \underline{y = \pm\sqrt{2}}$

$\downarrow$

$2y = -16y \rightarrow \underline{y=0}$

$\downarrow$

$\underline{x = \pm 2}$

Check: $(0,\sqrt{2})$ $(0,-\sqrt{2})$

$(2,0)$ $(-2,0)$

$f(0,\sqrt{2}) = f(0,-\sqrt{2}) = 2 \leftarrow$ max value of f

$f(2,0) = f(-2,0) = -16 \leftarrow$ min value of f.

MATH 213 - EXAM 2 - VERSION 2

1. $\vec{r}(t) = \langle \sin 2t, \cos 2t, 2t \rangle, \ t \geq 0$

$\vec{r}\,'(t) = \langle 2\cos 2t, -2\sin 2t, 2 \rangle$

$|\vec{r}\,'(t)| = (4\cos^2(2t) + 4\sin^2(2t) + 4)^{1/2}$

$= 8^{1/2} = 2\sqrt{2}.$

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle \frac{1}{\sqrt{2}}\cos 2t, \frac{-1}{\sqrt{2}}\sin 2t, \frac{1}{\sqrt{2}} \rangle$ //

$\vec{T}\,'(t) = \langle \frac{-2}{\sqrt{2}}\sin 2t, \frac{-2}{\sqrt{2}}\cos 2t, 0 \rangle$

$|\vec{T}\,'(t)| = (2\sin^2 2t + 2\cos^2(2t) + 0)^{1/2} = \sqrt{2}$

$K(t) = \dfrac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \dfrac{\sqrt{2}}{2\sqrt{2}} = \dfrac{1}{2}$ //

2. $f(x,y) = x^3 \sin(4xy)$

$$f_x = x^3 \cdot 4y \cos(4xy) + 3x^2 \sin(4xy)$$

$$= 4x^3 y \cos(4xy) + 3x^2 \sin(4xy) \mathbin{/\!/}$$

$$f_y = x^3 \cdot 4x \cos(4xy) = 4x^4 \cos(4xy) \mathbin{/\!/}$$

$$f_{yy} = 4x^4(-4x \sin(4xy)) = -16x^5 \sin(4xy) \mathbin{/\!/}$$

$$f_{xy} = f_{yx} = (4x^4)(-4y \sin(4xy)) + 16x^3 \cos(4xy)$$

$$= -16x^4 y \sin(4xy) + 16x^3 \cos(4xy) \mathbin{/\!/}$$

3. $f(x,y) = 3x^2y - 2y^2x$

a. $\nabla f = \langle 6xy - 2y^2, 3x^2 - 4xy \rangle$ //

b. $\nabla f(1,2) = \langle 4, -5 \rangle$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1, 3 \rangle}{\sqrt{10}} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$

$$D_{\vec{u}} f(1,2) = \langle 4, -5 \rangle \cdot \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle = \frac{-11}{\sqrt{10}}$$ //

c. MAX RATE OF CHANGE $= |\nabla f(1,2)|$

$$= |\langle 4, -5 \rangle| = (16+25)^{1/2} = \sqrt{41}$$ //

DIRECTION OF MAX R.o.C. $= \dfrac{\nabla f(1,2)}{|\nabla f(1,2)|}$

$$= \frac{\langle 4, -5 \rangle}{\sqrt{41}} = \left\langle \frac{4}{\sqrt{41}}, \frac{-5}{\sqrt{41}} \right\rangle$$ //

d. $L(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$

$$= -2 + 4(x-1) - 5(y-2)$$

$$= 4x - 5y + 4$$ //

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(e) $df = f_x(1,2)\,dx + f_y(1,2)\,dy$

$= 4(.1) - 5(.3) = -1.1$ //

4. $f(x,y) = x^3 + y^3 - 4x - 32y + 10$

$f_x = 3x^2 - 4 \quad f_y = 3y^2 - 32$

$3x^2 - 4 = 0 \rightarrow x = \pm \frac{2}{\sqrt{3}}$

$3y^2 - 32 = 0 \rightarrow y = \pm \frac{4\sqrt{2}}{\sqrt{3}}$

Crit pts: $\left(\frac{2}{\sqrt{3}}, \frac{4\sqrt{2}}{\sqrt{3}}\right) \left(\frac{2}{\sqrt{3}}, -\frac{4\sqrt{2}}{\sqrt{3}}\right) \left(\frac{-2}{\sqrt{3}}, \frac{4\sqrt{2}}{\sqrt{3}}\right)$

$\left(\frac{-2}{\sqrt{3}}, -\frac{4\sqrt{2}}{\sqrt{3}}\right)$ //

$f_{xx} = 6x \quad f_{yy} = 6y \quad f_{xy} = 0.$

$D(x,y) = 36xy.$

$D\left(\frac{2}{\sqrt{3}}, \frac{4\sqrt{2}}{\sqrt{3}}\right) > 0$

$f_{xx}\left(\frac{2}{\sqrt{3}}, \frac{4\sqrt{2}}{\sqrt{3}}\right) > 0.$

$\left(\frac{2}{\sqrt{3}}, \frac{4\sqrt{2}}{\sqrt{3}}\right)$ local minimum

$D\left(\frac{2}{\sqrt{3}}, \frac{-4\sqrt{2}}{\sqrt{3}}\right) < 0$

$\left(\frac{2}{\sqrt{3}}, \frac{-4\sqrt{2}}{\sqrt{3}}\right)$ saddle pt.

$D\left(\frac{-2}{\sqrt{3}}, \frac{4\sqrt{2}}{\sqrt{3}}\right) < 0$

$\left(\frac{-2}{\sqrt{3}}, \frac{4\sqrt{2}}{\sqrt{3}}\right)$ saddle pt

$D\left(\frac{-2}{\sqrt{3}}, \frac{-4\sqrt{2}}{\sqrt{3}}\right) > 0, \ f_{xx}\left(\frac{-2}{\sqrt{3}}, \frac{-4\sqrt{2}}{\sqrt{3}}\right) < 0.$

$\left(\frac{-2}{\sqrt{3}}, \frac{-4\sqrt{2}}{\sqrt{3}}\right)$ local maximum

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5. $f(x,y) = y^2 - 4x^2 \quad g(x,y) = x^2 + 2y^2 - 4$

$\nabla f = \langle -8x, 2y \rangle \quad \nabla g = \langle 2x, 4y \rangle$

$\nabla f = \lambda \nabla g \rightarrow -8x = 2\lambda x$

$2y = 4\lambda y.$

$-8x = 2\lambda x \rightarrow \underline{x=0} \quad \underline{\lambda = -4}$

$x^2 + 2y^2 = 4 \rightarrow \underline{y = \pm\sqrt{2}} \quad \downarrow$

$2y = -16y \rightarrow \underline{y=0}$

$x^2 + 2y^2 = 4 \longrightarrow x^2 = 4 \longrightarrow \underline{x = \pm 2}$

Check: $(0, \sqrt{2})$ $(0, -\sqrt{2})$ $(2, 0)$ $(-2, 0)$.

$f(0, \sqrt{2}) = f(0, -\sqrt{2}) = 2 \leftarrow$ max value of f

$f(2,0) = f(-2,0) = -16 \leftarrow$ min value of f.

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